A sufficient condition for univalence

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In 1941 S. OZAKI [3] proved the following:

Theorem (OZAKI) *Let* $f(z)=z+a_2z^2+\dots$ *be regular for* $|z|<1$. *If* $f(z)$ *satisfies in* $|z|<1$ *the condition*

$$\operatorname{Re}\left[1+z\frac{f''(z)}{f'(z)}\right]>-\frac{1}{2}, \tag{1}$$

then $f(z)$ *is univalent in* $|z|<1$.

T. UMEZAWA [4] in 1952 proved that (1) is also sufficient for the convexity of $f(z)$ in one direction.

In the present short note, besides generalizing Ozaki's theorem, we prove that the condition (1) implies that $f(z)$ is close-to-convex of order 1/2.

Theorem 1. *Let* $f(z)=z+a_2z^2+\dots$ *be regular for* $|z|<1$. *If for any two real numbers* λ *and* μ; $\lambda>0$, $f(z)$ *satisfies in* $|z|<1$ *condition:*

$$\operatorname{Re}\left[1+z\frac{f''(z)}{f'(z)}-(1-\lambda-i\mu)z\frac{f'(z)}{f(z)}\right]>\begin{cases}-\frac{\lambda}{2}, & 0<\lambda\leqq 1\\ -\frac{1}{2\lambda}, & \lambda>1,\end{cases} \tag{2}$$

then $f(z)$ *is univalent in* $|z|<1$.

PROOF. Let us put

$$A(z)=1+\frac{zf''(z)}{f'(z)}-(1-\lambda-i\mu)\frac{zf'(z)}{f(z)}$$

and first consider the case when $0<\lambda\leqq 1$. From the first inequalities in (2) we then conclude that the function:

$$\frac{A(z)+\frac{\lambda}{2}-i\operatorname{Im}A(0)}{\operatorname{Re}A(0)+\frac{\lambda}{2}}$$

is regular in $|z|<1$, assumes the value 1 at the origin, and has positive real part in $|z|<1$. Using the expression for $A(z)$, we can therefore write

$$\frac{\lambda}{2}+1+\frac{zf''(z)}{f'(z)}-(1-\lambda-i\mu)\frac{zf'(z)}{f(z)} = \frac{3}{2}\lambda P(z)+i\mu, \tag{3}$$

where $P(z)=1+p_1 z+p_2 z^2+\dots$ is regular in $|z|<1$ and has positive real part therein.

If in the right side of (3) for the function $P(z)$, we use the following well known Herglotz' representation:

$$P(z) = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, d\mu(t),$$

where $\mu(t)$ is a non-decreasing function defined for $-\pi \leqq t \leqq \pi$, with total variation equal to 1, that is,

$$\int_{-\pi}^{\pi} d\mu(t) = 1,$$

we obtain, after an integration, the following relation:

$$\log\left\{\frac{z^{1-\lambda-i\mu}f'(z)}{f(z)^{1-\lambda-i\mu}}\right\} = -3\lambda \int_{-\pi}^{\pi} \log(1-ze^{-it})\, d\mu(t). \tag{4}$$

(Here principal values are understood to have been taken)

We now define a function $g(z)$ as follows:

$$\lambda \log \frac{g(z)}{z} = -2\lambda \int_{-\pi}^{\pi} \log(1-ze^{it})\, d\mu(t). \tag{5}$$

It is readily seen that the function $g(z)$ is starlike in $|z|<1$.

From (4) and (5) we obtain

$$\log\left\{\frac{z^{1-i\mu}f'(z)}{f(z)^{1-\lambda-i\mu}g(z)^{\lambda}}\right\} = -\lambda \int_{-\pi}^{\pi} \log(1-ze^{-it})\, d\mu(t). \tag{6}$$

It is a simple matter to verify that the function $\log(1-ze^{-it})$ maps $|z|<1$ onto a convex domain and therefore we conclude that $-\lambda \int_{-\pi}^{\pi} \log(1-ze^{-it})\, d\mu(t)$ is subordinate to $\log(1-ze^{-it})^{-\lambda}$ in $|z|<1$ (For the *definition of subordination*, see [2; p 227]). In symbols

$$-\lambda \int_{-\pi}^{\pi} \log(1-ze^{-it})\, d\mu(t) \prec \log\frac{1}{(1-ze^{-it})^{\lambda}}, \quad |z|<1. \tag{7}$$

From (6) and (7) we then have

$$\frac{z^{1-i\mu}f'(z)}{f(z)^{1-\lambda-i\mu}g(z)^{\lambda}} \prec \frac{1}{(1-ze^{-it})^{\lambda}} \quad \text{in} \quad |z|<1. \tag{8}$$

Since $0<\lambda\leqq 1$, from (8) we deduce that

$$\operatorname{Re}\left[\frac{z^{\lambda-i\mu}f'(z)}{f(z)^{1-\lambda-i\mu}g(z)^{\lambda}}\right]>0$$

in $|z|<1$ and therefore $f(z)$ is BAZILEVIČ [1] and consequently univalent in $|z|<1$. The case $\lambda>1$ may be similarly handled. This complets the proof of Theorem 1.

If we take $\lambda=1$ and $\mu=0$ in Theorem 1, we obtain Ozaki's theorem.

Theorem 2. *If* $f(z)$ *satisfies* (1) *in* $|z|<1$, *then* $f(z)$ *is close-to-convex of order* 1/2 *in* $|z|<1$.

PROOF. Taking $\lambda=1$ and $\mu=0$ in (8) we find that

$$\frac{zf'(z)}{g(z)}\prec\frac{1}{1-z^{-it}}\quad(|z|<1)$$

from which we deduc tehat $\operatorname{Re}\{zf'(z)|g(z)\}>1/2$ in $|z|<1$ and therefore $f(z)$ is close-to-convex of order 1/2 in $|z|<1$.

References

[1] I. E. BAZILEVIČ, On a case of integrability in quadratures of the Loewner—Kufarev equation *Mat. Sb.* **37** (79) (1955), 471—476. (Russian.)

[2] Z. NEHARI, Conformal Mapping, *New York*, 1952.

[3] S. OZAKI, On the theory of multivalent functions II, sci, *Rep. Tokyo Bunrika Daigaku*, **4** (1941), 45—86.

[4] T. UMEZAWA, Analytic functions convex in one direction, *J. Math. Soc.* JAPAN, **4** (1952), 194—202.

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