Publ. Math. Debrecen
50 / 1-2 (1997), 29–36

On the form of some solutions of a linear iterative functional inequality of n-th order

By BOGDAN CHOCZEWSKI (Kraków) and MARIA STOPA (Kraków)

Abstract. Three representation theorems are proved for continuous solutions of a linear iterative functional inequality (1) of n-th order, with constant coefficients.

1. Introduction

We consider the following functional inequality with constant coefficients:

$$\psi_1(f^n(x)) + a_{n-1}\psi_1(f^{n-1}(x)) + \ldots + a_0\psi_1(x) \le 0 \tag{1}$$

where ψ_1 is an unknown function and f^i denotes the i-th iterate of a given function f. The paper contains three theorems of the representation type for the inequality (1) concerning its continuous solutions. They are based on results from paper [3] obtained for the inequality of second order of type (1) (with some special functional coefficients).

We assume the following hypotheses:

(H) $f : I \to I,\ I = [0, a),\ a > 0,\ f$ *is continuous and strictly increasing on* I *and* $0 < f(x) < x,\ x \in I \setminus \{0\};\ a_i \in \mathbb{R},\ i = 0, \ldots, n-1$.

The polynomial:

$$w_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 \tag{2}$$

is called the characteristic polynomial of the inequality (1). Denote by $\lambda_1, \ldots, \lambda_n$ the roots of this polynomial.

If $\psi_1 :\to \mathbb{R}$ is a solution of inequality (1), then the function $\psi_2 : I \to \mathbb{R}$, given by the formula

$$\psi_2(x) := \psi_1(f(x)) - \lambda_1\psi_1(x) \tag{3}$$

satisfies the inequality of $(n-1)$st order:

$$\psi_2(f^{n-1}(x)) + b_{n-2}\psi_2(f^{n-2}(x)) + \ldots + b_0\psi_2(x) \le 0, \quad x \in I \tag{4}$$

where $b_0, \ldots, b_{n-2}$ are the coefficients of the polynomial obtained on dividing polynomial (2) by $(\lambda - \lambda_1)$:

$$(\lambda - \lambda_1)(\lambda^{n-1} + b_{n-2}\lambda^{n-2} + \ldots + b_0) = w_n(\lambda).$$

In turn, introducing the functions $\psi_i : I \to \mathbb{R}$:

$$\psi_i(x) = \psi_{i-1}(f(x)) - \lambda_{i-1}\psi_{i-1}(x), \quad (i = 3, \ldots, n)$$

we get the following system

$$\left\{\begin{array}{lcl} \psi_1(f(x)) - \lambda_1\psi_1(x) & = & \psi_2(x) \\ & \vdots & \\ \psi_{n-1}(f(x)) - \lambda_{n-1}\psi_{n-1}(x) & = & \psi_n(x) \\ \psi_n(f(x)) - \lambda_n\psi_n(x) & \le & 0 \end{array}\right. \tag{5}$$

which is equivalent to inequality (1), in the following sense: if ψ_1 is a solution of (1) then the system of functions $(\psi_1, \ldots, \psi_n)$ satisfies (5); and conversely, given a solution $(\psi_1, \ldots, \psi_n)$ of system (5), the function ψ_1 satisfies inequality (1).

2. Preliminaries

In this paper we shall consider the following two cases concerning the roots of the characteristic polynomial (2)

(I) $$1 > |\lambda_i| > \lambda_n > 0,\ i = 1, \ldots, n-1; \quad \lambda_1 \in \mathbb{R}$$

(II) $$\begin{gathered} |\lambda_1| \ge |\lambda_2| \ge \ldots \ge |\lambda_k| > 1, \quad |\lambda_{k+1}| = \ldots = |\lambda_{k+l}| = 1 \\ 1 > |\lambda_i| > \lambda_n > 0,\ i = k+l+1, \ldots, n-1;\ \lambda_{k+l+1} \in \mathbb{R} \\ 1 \le k+l < n,\ k, l \in \{1, \ldots, n-1\}. \end{gathered}$$

Note that in both cases complex roots of (2) are allowed and that in the case (II), we practically classify only the roots according to whether they are either outside, or on, or inside the unit circle.
In the sequel we shall use the following notions and notations.
1. A function $\eta : I \to \mathbb{R}$ is said to be *f-decreasing* iff

$$\eta(f(x)) \le \eta(x), \quad x \in I$$

2. For a given solution $\psi_1 : I \to \mathbb{R}$ of (1) let $\psi_n : I \to \mathbb{R}$ be the function *corresponding to* (defined by) this ψ_1 via system (5). By $\mathbf{\Psi}_n$ we denote the family of solutions to (1), which are asymptotically comparable (at the origin) with a solution $\tilde{\phi} : I \to \mathbb{R}$ of the Schröder equation

$$\phi(f(x)) = \lambda_n \phi(x), \quad x \in I. \tag{6}$$

The definition of $\mathbf{\Psi}_n$ follows:

$$\mathbf{\Psi}_n = \Big\{\psi_1 : I \to \mathbb{R};\ \psi_n \geq 0 \text{ in } I \text{ and there exists a } \tilde{\phi} : I \to \mathbb{R} \text{ satisfying (6)}$$
$$\tilde{\phi}(x) > 0 \text{ in } I \setminus \{0\} \text{ such that } \lim_{x\to 0} \tfrac{\psi_1(x)}{\tilde{\phi}(x)} \neq 0\Big\}.$$

3. Case $n = 2$

The case of $n = 2$ in (1) is a special case of the inequality that has been considered in [3]

$$\psi_1(f^2(x)) - (\lambda(f(x)) + \mu(x))\, \psi_1(f(x)) + \lambda(x)\mu(x)\psi_1(x) \leq 0 \tag{7}$$

where ψ_1 is an unknown function, μ, λ and f are given functions, f^2 denotes the second iterate of the function f. Indeed, putting in (7) $\lambda(x) \equiv \lambda_1$, $\mu(x) \equiv \lambda_2$ where λ_1, and λ_2 are the roots of the polynomial ($w_2(\lambda) = \lambda^2 + a_1\lambda + a_0$), we obtain (1) with $n = 2$. In what follows we shall make use of the following results adapted from [3] to the case of $n = 2$ in (1).

Lemma 1. *Assume* (H) *to hold and let* $\phi_2 : I \to \mathbb{R}^+$ *be a continuous solution of equation* (6) *(with* $n = 2$*) and let*

$$(i) \qquad 0 < \lambda_2 < |\lambda_1| < 1.$$

Then each function:

$$\psi_1(x) := -\phi_2(x) \sum_{i=0}^{\infty} \frac{\eta(f^i(x))}{\lambda_1^{i+1}} \lambda_2^i, \quad x \in I \tag{8}$$

(where $\eta : I \to \mathbb{R}$ *is a continuous,* $\{f\}$*-decreasing function) is a continuous solution of the inequality* (1) *(with* $n = 2$*).*

Lemma 2. *If* (H) *and* (i) *hold and* $\psi_1 \in \mathbf{\Psi}_n$ *is a solution of* (1)*, then*

(a) There exists the function $\phi_0 : I \to \mathbb{R}$ *given by:*

$$\phi_0(x) = \lim_{j\to\infty} \left[\psi_2(f^j(x))\lambda_2^{-j}\right], \quad x \in I \tag{9}$$

where ψ_2 is the function defined by (3), *and this function is continuous in I, satisfies equation* (6) $(n = 2)$ *and* $\phi_0(x) > 0$ *in* $I \setminus \{0\}$.

(b) There exists the limit:

$$\lim_{x \to 0} [\psi_1(x)/\phi_0(x)] = (\lambda_2 - \lambda_1)^{-1} \tag{10}$$

where ϕ_0 is given by (9).

(c) If $\psi_1 \in \mathbf{\Psi}_2$ is a solution of (1) *(with $n = 2$), then there is exactly one $\{f\}$-decreasing and continuous function $\eta : I \to \mathbb{R}^+$, $\eta(0) = 1$, such that ψ_1 is given by* (8) *(with ϕ_2 replaced by $\phi_0(x)$, defined by* (9)*).*

4. Case (I)

We start with the case (I) of all the characteristic roots of (2) in absolute value less than unity. The following theorem corresponds to Lemma 1.

Theorem 1. *Assume* (H) *and* (I). *If $\phi_n : I \to \mathbb{R}$ is a nonnegative solution of* (6) *then every function $\psi_1 : I \to \mathbb{R}$ given by:*

$$\psi_1(x) = (-1)^{n+1} \phi_n(x) \sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1} \cdot \ldots \cdot \lambda_{n-1}^{i_{n-1}+1}} \lambda_n^{i_1+\ldots+i_{n-1}}, \quad x \in I \tag{11}$$

(where $\eta : I \to \mathbb{R}$ is a continuous, $\{f\}$-decreasing function) is a continuous solution of inequality (1).

Proof. To start the inductive proof let $n = 2$. Then the Theorem reduces to Lemma 1, cf. (8).

Assume the assertion is valid for inequalities (1) of order $n - 1$. To prove that it is also for inequalities of n-th order, it is enough to check (cf. Section 2) that the function $\psi_2(x) : I \to \mathbb{R}$ given by (3) satisfies inequality (4). We calculate

$$\begin{aligned}
\psi_2(x) &= \psi_1(f(x)) - \lambda_1 \psi_1(x) \\
&= (-1)^{n+1} \Bigg[\phi_n(f(x)) \sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \frac{\eta(f^{i_1+\ldots+i_{n-1}+1}(x))}{\lambda_1^{i_1+1} \cdot \ldots \cdot \lambda_{n-1}^{i_{n-1}+1}} \lambda_n^{i_1+\ldots+i_{n-1}} \\
&\qquad - \lambda_1 \phi_n(x) \sum_{i_1=0}^{\infty} \cdots \sum_{i_{n-1}=0}^{\infty} \frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1} \cdot \ldots \cdot \lambda_{n-1}^{i_{n-1}+1}} \lambda_n^{i_1+\ldots+i_{n-1}} \Bigg]
\end{aligned}$$

$$= (-1)^{n+1}\Bigg[\phi_n(x)\sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\lambda_1\frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_1+\ldots+i_{n-1}}$$

$$-\phi_n(x)\sum_{i_2=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\frac{\eta(f^{i_2+\ldots+i_{n-1}}(x))}{\lambda_2^{i_2+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_2+\ldots+i_{n-1}}$$

$$-\phi_n(x)\sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\lambda_1\frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_1+\ldots+i_{n-1}}\Bigg]$$

$$= (-1)^n\phi_n(x)\sum_{i_2=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\frac{\eta(f^{i_2+\ldots+i_{n-1}}(x))}{\lambda_2^{i_2+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_2+\ldots+i_{n-1}}$$

since the first and the last expression in brackets cancel each other. Thus the function ψ_2 is given by (11) (with n replaced by $n-1$), therefore, by the induction hypothesis, it satisfies inequality (4). According to (3), the function ψ_1 satisfies inequality (1).

To proceed further, we need the Lemma corresponding to Lemma 2 (a) and (b). This is our

Lemma 3. *Assume* (H) *and* (I). *If $\psi_1 \in \mathbf{\Psi}_n$, then there exist:*
– the function (with ψ_n generated in (5) *by ψ_1)*

$$\phi_0(x) = \lim_{i\to\infty}[\psi_n(f^i(x))\lambda_n^{-i}], \quad x \in I \tag{12}$$

which satisfies equation (6), *is continuous in I and positive in $I \setminus \{0\}$.*
– the limit

$$\lim_{x\to 0}[\psi_1(x)/\phi_0(x)] = \prod_{i=1}^{n-1}(\lambda_n - \lambda_i)^{-1} \tag{13}$$

Proof. For $n = 2$ our Lemma is just Lemma 2 (a) and (b).

Assume the Lemma to be true for inequalities (1) of order $n-1$, $n > 2$. Since ψ_1 satisfies (1), it satisfies also the equation

$$\psi_1(f(x)) - \lambda_1\psi_1(x) = \psi_2(x) \tag{14}$$

where ψ_2 is a solution to (4). From the definition of the family $\mathbf{\Psi}_n$ (cf. Section 2 — the limit condition) we conclude, by (14), that there exists $\lim_{x\to 0}\psi_2(x)/\tilde{\phi}(x)$ and it does not vanish. Thus $\psi_2 \in \mathbf{\Psi}_{n-1}$. By the induction hypothesis we get the existence of the following limits:

$$\lim_{i\to\infty}[\lambda_n^{-i}\psi_n(f^i(x))] = \phi_0(x), \quad x \in I$$

(this is so since ψ_n is generated by ψ_2 as well) and

$$\lim_{x\to 0}[\psi_2(x)/\phi_0(x)] = \prod_{i=2}^{n-1}(\lambda_n - \lambda_i)^{-1}. \tag{15}$$

Moreover, the function ϕ_0 actually has all the required properties. Hence, as $\psi_1 \in \mathbf{\Psi}_n$, there exists the limit $\lim_{x\to 0}[\psi_1(x)/\phi_0(x)]$. To calculate it, divide first both sides of (14) by $\phi_0(f(x))$, $(x \in I \setminus \{0\})$:

$$\frac{\psi_1(f(x))}{\phi_0(f(x))} - \frac{\lambda_1}{\lambda_n}\frac{\psi_1(x)}{\phi_0(x)} = \frac{\psi_2(x)}{\lambda_n \phi_0(x)}.$$

Passing to the limit, as $x \to 0$, here, we obtain

$$\lim_{x\to 0}\frac{\psi_1(x)}{\phi_0(x)}\left[1 - \frac{\lambda_1}{\lambda_n}\right] = \frac{1}{\lambda_n}\lim_{x\to 0}\frac{\psi_2(x)}{\phi_0(x)}.$$

On account of (15) we have (13).

For the functions belonging to the family $\mathbf{\Psi}_n$, we have the following representation theorem.

Theorem 2. *Assume* (H) *and* (I). *Let* $\psi_1 \in \mathbf{\Psi}_n$. *There is exactly one function* $\eta : I \to \mathbb{R}$, *continuous,* $\{f\}$*-decreasing,* $\eta(0) = 1$, *such that*

$$\psi_1(x) = (-1)^{n+1}\phi_0(x)\sum_{i_1=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_1+\ldots+i_{n-1}},$$
$$x \in I \tag{16}$$

where ϕ_0 *is defined by* (12).

Proof. For $n = 2$ this is Lemma 2 (c).

Assume our Theorem to be true for inequalities (1) of order $n-1$. Since ψ_1 satisfies (1) (cf. Section 2 — the definition of $\mathbf{\Psi}_n$), it also satisfies equation (14), where ψ_2 fulfils (4). This means that $\psi_2 \in \mathbf{\Psi}_{n-1}$ and, by the induction hypothesis, there is exactly one function $\eta : I \to \mathbb{R}$, which is continuous, $\{f\}$-decreasing, $\eta(0) = 1$, such that

$$\psi_2(x) = (-1)^{n}\phi_0(x)\sum_{i_2=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\frac{\eta(f^{i_2+\ldots+i_{n-1}}(x))}{\lambda_2^{i_2+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_2+\ldots+i_{n-1}},$$
$$x \in I. \tag{17}$$

From Lemma 3 we know that (13) holds. Thus the function

$$\widehat{\psi}_1(x) = \begin{cases} \dfrac{\psi_1(x)}{\phi_0(x)}, & x \in I \setminus \{0\} \\ \prod_{i=1}^{n-1} (\lambda_n - \lambda_i)^{-1}, & x = 0 \end{cases} \tag{18}$$

is continuous in I. The function $\phi = \widehat{\psi}_1$ satisfies the equation

$$\phi(f(x)) - \frac{\lambda_1}{\lambda_n}\phi(x) = \frac{\psi_2(x)}{\lambda_n \phi_0(x)}.$$

By (I) we have $\left|\frac{\lambda_1}{\lambda_n}\right| > 1$, so that this linear iterative functional equation possesses the unique continuous solution which is given by the formula

$$\widehat{\psi}_1(x) = -\frac{1}{\lambda_n}\sum_{i=0}^{\infty} \frac{\psi_2(f^i(x))(\frac{\lambda_n}{\lambda_1})^{i+1}}{\phi_0(f^i(x))}$$

(cf. [1] Th. 2.7 or [2] Ch. 3.1C). According to (17) and (16) formula (16) follows.

5. Case (II)

We conclude the paper with a theorem concerning the case (II). Let

$$m = n - (k + l)$$

be the number of those characteristic roots of the polynomial (2) which are inside the unit circle.

Theorem 3. *Assume* (H) *and* (II). *If* $\psi_1 : I \to \mathbb{R}$ *is a continuous solution of the inequality* (1) *such that the function* ψ_{k+l+1}, *generated by* ψ_1 *via the system* (5), *belongs to the family* $\mathbf{\Psi}_m$ *and if at least one root of the characteristic polynomial* (2) *is equal one, then there exist a constant* $c \in \mathbb{R}$ *and a continuous,* $\{f\}$*-decreasing function* $\eta : I \to \mathbb{R}$, $\eta(0) = 1$, *such that*

$$\psi_1(x) = \frac{c}{\prod_{i=1}^{k+r}(1-\lambda_i)} \tag{19}$$

$$+(-1)^{n+1}\phi_0(x)\sum_{i_1=0}^{\infty}\cdots\sum_{i_{n-1}=0}^{\infty}\frac{\eta(f^{i_1+\ldots+i_{n-1}}(x))}{\lambda_1^{i_1+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{n-1}+1}}\lambda_n^{i_1+\ldots+i_{n-1}}, \quad x \in I$$

where ϕ_0 *is given by* (12) *and* r *denotes the number of those characteristic roots of* (2) *which are from the unit circle and differ from* 1 *and* -1.

Proof. If $\psi_1 : I \to \mathbb{R}$ is a continuous solution of inequality (1) such that $\psi_{k+l+1} \in \mathbf{\Psi}_m$, then by Theorem 2 the function ψ_{k+l+1} is given by the formula

$$\psi_{k+l+1}(x) = (-1)^{m+1}\phi_0(x)\sum_{i_1=0}^{\infty}\dots\sum_{i_{m-1}=0}^{\infty}\frac{\eta(f^{i_1+\dots+i_{m-1}}(x))}{\lambda_{k+l+1}^{i_1+1}\cdot\ldots\cdot\lambda_{n-1}^{i_{m-1}+1}}\lambda_n^{i_1+\dots+i_{m-1}},$$

$$x \in I$$

Using this formula in the $(k+l)$-th equation of the system (5), i.e.

$$\psi_{k+l}(f(x)) - \lambda_{k+l}\psi_{k+l}(x) = \psi_{k+l+1}(x)$$

we may go back to the first equation of (5) by the same way as it has been done in the proof of Theorem 2 from [4]. This procedure, whose details are not reproduced here, then yields formula (19).

References

[1] M. Kuczma, Functional equations in a single variable, Monografie Mat. 46, *Polish Scientific Publishers (PWN), Warszawa*, 1968.

[2] M. Kuczma, B. Choczewski and R. Ger, Iterative Functional Equations, Encyclopedia of Mathematics and Its Applications 32, *Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, Sydney*, 1990.

[3] M. Stopa, On the form of solutions of some iterative functional inequality, *Publ. Math. Debrecen* **45** (1994), 371–377.

[4] M. Stopa, On the form of solutions of the iterative functional equation of the n-th order, *Demonstratio Math.* **28** (1995), 625–634.

BOGDAN CHOCZEWSKI
INSTITUTE OF MATHEMATICS WSP
PODCHORĄŻYCH 2
30–084 KRAKÓW
POLAND

MARIA STOPA
INSTITUTE OF MATHEMATICS AGH
AL. MICKIEWICZA 30
30–059 KRAKÓW
POLAND

(Received May 18, 1995)