The determination of the jump of a function by Nörlund means

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1. Let f be a 2π-periodic function integrable in the sense of Lebesgue and let

$$\text{(A)} \qquad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. The series conjugate to (A) is

$$\text{(B)} \qquad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x)$$

whose n-th partial sum will be denoted by $\tilde{S}_n(x)$. If there exists a $D(x)$ such that

$$\psi(t) = f(x+t) - f(x-t) - D(x) = o(1) \quad (t \to 0),$$

then $D(x)$ is called the *jump* of the function f at the point x. Even when this does not hold, $D(x)$ may satisfy either the relation

$$\Psi(t) = \int_0^t |\psi(u)|\, du = o(t) \quad (t \to 0),$$

or, still more generally, the relation

$$\psi_1(t) = \int_0^t \psi(u)\, du = o(t) \quad (t \to 0).$$

In each of these latter cases, f will be said to have a generalized jump $D(x)$ at the point x.

The problem of the determination of the jump of f by means of its Fourier coefficients has been investigated by a number of writers. Broadly speaking, there are two different methods that have been adopted in this connection. FEJÉR [6], CSILLAG [4], SZIDON [18], LUKÁCS [9], ZYGMUND [19], and SZÁSZ [15] determined the jump by considering the summability of the sequence $\{nB_n\}$. Subsequently SZÁSZ [16] devised a new method for the determination of the jump. He considered the difference of the means of two different orders of the partial sums of the corresponding conjugate series instead of the summability of the sequence $\{nB_n\}$. This

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line of research has been pursued further by MARUYAMA [10], CHOW [2], [3], MOHANTY [13], MINAKSHISUNDARAM [11], SZÁSZ [16], [17], MISRA [12] and others.

In this paper, we obtain analogous results using Nörlund's method of summability.

2. Let $\{p_n\}$ be a sequence of complex numbers such that $P_n = p_0 + \ldots + p_n \neq 0$. The series $\sum_0^\infty u_n$ with partial sum $s_n = \sum_0^n u_k$ is said to be summable by the Nörlund's method of summation defined by the sequence $\{p_n\}$, or simply summable (N, p) if

$$\lim_{n\to\infty} N_n^p(s) = \lim_{n\to\infty} P_n^{-1} \sum_{k=0}^n p_{n-k} s_k = \lim_{n\to\infty} P_n^{-1} \sum_{k=0}^n P_{n-k} u_k$$

exists. The conditions for regularity are

$$\text{(i)}\ \ p_n = o(P_n) \qquad (n \to \infty),$$

$$\text{(ii)}\ \ \sum_{k=0}^n |p_k| = O(P_n) \quad (n \to \infty),$$

of which the latter is automatically satisfied when the sequence $\{p_n\}$ is positive.

We write throughout

$$\Delta^1 p_k = \Delta p_k = p_k - p_{k-1}, \quad \Delta^r p_k = \Delta(\Delta^{r-1} p_k) \quad (r > 1)$$

with $p_k = 0$ for $k < 0$.

We prove the following theorems.

Theorem 1. *If there exists a $D(x)$ such that $\Psi(t) = o(t)$ $(t \to 0)$ and, in particular, if $\psi(t) = 0(1)$ $(t \to 0)$, the sequence $\{nB_n(x)\}$ is summable (N, p) to $D(x)/\pi$, provided*

$$\text{(i)}\ \ \sum_{k=1}^n k|\Delta^2 p_k| = O\left(\frac{P_n}{n}\right),$$

$$\text{(ii)}\ \ \sum_{k=1}^n \frac{|P_k|}{k^2} = O\left(\frac{P_n}{n}\right).$$

Theorem 2. *If there exists a $D(x)$ such that $\psi_1(t) = o(t)$ $(t \to 0)$, the sequence $\{nB_n(x)\}$ is summable (N, p) to $D(x)/\pi$, provided*

$$\text{(i)}\ \ \sum_{k=1}^n k|\Delta^3 p_k| = O\left(\frac{P_n}{n^2}\right),$$

$$\text{(ii)}\ \ \sum_{k=1}^n \frac{|P_k|}{k^3} = O\left(\frac{P_n}{n^2}\right).$$

Theorem 3. *If there exists a $D(x)$ such that $\Psi(t) = o(t)$ $(t \to 0)$ and if $\frac{m}{n} \to d$ as $n \to \infty$, then for any regular (N, p):*

$$\lim_{n\to\infty} [N_m^p\{\tilde{S}(x)\} - N_n^p\{\tilde{S}(x)\}] = \frac{D(x)}{\pi} \log d,$$

provided that there exists an $l \neq 0$ *such that*

$$\frac{np_n}{P_n} = l + \frac{A}{n} + o\left(\frac{1}{n}\right) \quad (n \to \infty)$$

for some A.

We remark here that, as shown in § 3, the different hypotheses on the sequence $\{p_n\}$ made in theorems 1, 2 and 3 imply that (N, p) is regular.

If, in the above theorems, we choose

$$p_n = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \quad (\alpha > -1),$$

we get, as corollaries, the following theorems due to Fejér [6] and Chow [2] (cf. also Szász [16], Maruyama [10]).

Theorem 1′. *If there exists a* $D(x)$ *such that* $\Psi(t) = o(t)$ $(t \to 0)$ *and, in particular, if* $\psi(t) = o(1)$ $(t \to 0)$, *the sequence* $\{nB_n(x)\}$ *is summable* (C, α) *to* $D(x)/\pi$ *for every* $\alpha > 1$.

Theorem 2′. *If there exists a* $D(x)$ *such that* $\psi_1(t) = o(t)$ $(t \to 0)$, *the sequence* $\{nB_n(x)\}$ *is summable* (C, α) *to* $D(x)/\pi$ *for every* $\alpha > 2$.

Theorem 3′. *If there exists a* $D(x)$ *such that* $\Psi(t) = o(t)$ $(t \to 0)$ *and if* $\frac{m}{n} \to d$ *as* $n \to \infty$, *then*

$$\lim_{n \to \infty} \{\tilde{S}_m^\alpha(x) - \tilde{S}_n^\alpha(x)\} = \frac{D(x)}{\pi} \log d$$

for every $\alpha > 0$, *where* $S_n^\alpha(x)$ *denotes the* (C, α) — *means of the sequence* $\tilde{S}_n(x)$.

3. In order to prove the theorems we need the following lemmas whose proofs, if not detailed below, are known (cf. [1], [5], [7], [8]).

Lemma 0. *For a given sequence* $\{s_n\}$, *let* S_n^r *be defined inductively by* $S_n^0 = s_n$, $S_n^r = S_0^{r-1} + S_1^{r-1} + \dots + S_n^{r-1}$ $(r \geqq 1)$. *Let* λ *be any real number, and* $\{T_n\}$ *a sequence of positive numbers. Suppose that, for some* $r \geqq 1$

$$\sum_{k=1}^{n} k^\lambda |S_k^{r+1}| \leqq KT_n$$

and

$$\sum_{k=1}^{n} k^{\lambda+r+1} |s_k| \leqq KT_n.$$

Then, for $1 \leqq \varrho \leqq r$ (ϱ *an integer*)

$$\sum_{k=1}^{n} k^{\lambda+r+1-\varrho} |S_k^\varrho| \leqq KT_n.$$

Lemma 1. *If* $\sum_{k=1}^{n} k|\Delta p_k| = O(P_n)$, *then* $np_n = O(P_n)$ *and* $\sum_{k=0}^{n} |p_k| = O(P_n)$.

Lemma 2. *If* (i) $\sum_{k=1}^{n} k|\Delta p_k| = O(P_n)$ *and* (ii) $|P_n| \to \infty$, *then* $\sum_{k=1}^{n} |\Delta p_k| = o(P_n)$.

The result holds, if, in particular, (ii) is replaced by (ii)′ $\sum_{k=1}^{n} \frac{|P_k|}{k} = O(P_n)$ which implies the former.

Lemma 3. *If* $\sum_{k=0}^{n} |\Delta p_k| = o(P_n)$, *then* $\sum_{k=0}^{n} p_k e^{ikt} = o(P_n)$ *uniformly in* t *for* $0 < \delta \leqq |t| \leqq \pi$.

Lemma 4. *If*

$$(*) \qquad \sum_{k=1}^{n} \frac{|P_k|}{k^j} = O\left(\frac{P_n}{n^{j-1}}\right) \quad (j \geqq 1),$$

then $n^{j-1} = o(P_n)$ *and*

$$\sum_{k=1}^{n} \frac{|P_k|}{k^{j+1}} = o\left(\frac{P_n}{n^{j-1}}\right).$$

PROOF. The condition $(*)$ for all n, implies that for some constant $A > 0$

$$n^{j-1} \leqq A|P_1|^{-1}|P_n| \quad \text{so that} \quad A^{-1}|P_1| \sum_{1}^{n} \frac{1}{k} \leqq A \frac{|P_n|}{n^{j-1}} \quad \text{and} \quad n^{j-1} = o(P_n).$$

Also, for every $n \geqq N$,

$$\frac{n^{j-1}}{|P_n|} \sum_{k=1}^{n} \frac{|P_k|}{k^{j+1}} \leqq \frac{n^{j-1}}{|P_n|} \sum_{1}^{N} \frac{|P_k|}{k^{j+1}} + \frac{1}{N} \frac{n^{j-1}}{|P_n|} \sum_{N+1}^{n} \frac{|P_k|}{k^j} \leqq \frac{n^{j-1}}{|P_n|} \sum_{1}^{N} \frac{|P_k|}{k^{j+1}} + \frac{A}{N}$$

so that

$$\varlimsup_{n\to\infty} \frac{n^{j-1}}{|P_n|} \sum_{k=1}^{n} \frac{|P_k|}{k^{j+1}} \leqq \frac{A}{N}.$$

Since N is arbitrary, it follows that $\sum_{k=1}^{n} \frac{|P_k|}{k^{j+1}} = o\left(\frac{P_n}{n^{j-1}}\right)$.

Lemma 5. *If* $\sum_{k=1}^{n} k|\Delta^2 p_k| = O\left(\frac{P_n}{n}\right)$ *and* $\sum_{k=1}^{n} \frac{|P_k|}{k^2} = O\left(\frac{P_n}{n}\right)$, *then*

$$\sum_{k=1}^{n} |\Delta p_k| = O\left(\frac{P_n}{n}\right) \quad \text{and} \quad \sum_{k=1}^{n} |\Delta^2 p_k| = O\left(\frac{P_n}{n}\right).$$

PROOF. The first assertion follows from lemma 0, if we choose $r=2$, $\lambda=-2$ and $s_k = \Delta^2 p_k$ so that $S_k^1 = \Delta p_k$, $S_k^2 = p_k$, $S_k^3 = P_k$. Using lemma 4, we get

$$\sum_{k=1}^{n} |\Delta^2 p_k| = \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \sum_{\nu=1}^{k} \nu|\Delta^2 p_\nu| + \frac{1}{n} \sum_{\nu=1}^{n} \nu|\Delta^2 p_\nu| = O\left(\sum_{k=1}^{n} \frac{P_k}{k^3}\right) + O\left(\frac{P_n}{n^2}\right) =$$

$$= o\left(\frac{P_n}{n}\right) + O\left(\frac{P_n}{n^2}\right) = o\left(\frac{P_n}{n}\right).$$

Lemma 6. *If*

$$p_n = o(P_n), \quad \sum_{k=0}^{n} |\Delta p_k| = o(P_n)$$

and $\sum_{k=0}^{n}(n-k)|\Delta^2 p_k| = o(P_n)$, *then* $\sum_{k=0}^{n}(n-k)p_k e^{ikt} = o(P_n)$ *uniformly in t for* $0<\delta\leqq|t|\leqq\pi$.

Lemma 7. *If*

$$\sum_{k=1}^{n} k|\Delta^3 p_k| = O\left(\frac{P_n}{n^2}\right) \quad \text{and} \quad \sum_{k=1}^{n}\frac{|P_k|}{k^3} = O\left(\frac{P_n}{n^2}\right),$$

then $\sum_{=1}^{n}\frac{|\Delta p_k|}{k} = O\left(\frac{P_n}{n^2}\right)$.

Proof. Apply lemma 0 with $s_k=\Delta^3 p_k$, $r=3$ and $\lambda=-3$.

Lemma 8. *If*

$$\sum_{k=1}^{n} k|\Delta^3 p_k| = O\left(\frac{P_n}{n^2}\right) \quad \text{and} \quad \sum_{k=1}^{n}\frac{|P_k|}{k^3} = O\left(\frac{P_n}{n^2}\right)$$

then $\sum_{k=1}^{n} k|\Delta p_k| = O(P_n)$ *and* $\sum_{k=1}^{n}|\Delta^3 p_k| = o\left(\frac{P_n}{n^2}\right)$.

Proof. The proof is similar to that of lemma 5.

Lemma 9. *If*

$$p_n = o(P_n) \quad \text{and} \quad \sum_{k=0}^{n}(n-k)^{j-1}|\Delta^j p_k| = o(P_n) \quad (j = 1, 2, 3),$$

then $\sum_{k=0}^{n}(n-k)^2 p_k e^{ikt} = o(P_n)$ *uniformly in t for* $0<\delta\leqq|t|\leqq\pi$.

Proof. Making three successive Abel's transformations, we get

$$\sum_{k=0}^{n}(n-k)^2 p_k e^{ikt} =$$

$$= \sum_{k=0}^{n}(1-e^{it})^{-3}[\{(n-k)^2(\Delta^3 p_k)-6(n-k)(\Delta^2 p_{k-1})-3\Delta^2 p_{k-1}+6\Delta p_{k-2}\}e^{ikt}+$$

$$+(1-e^{it})p_{n-1}e^{i(n+1)t}+2(\Delta p_{n-1}-p_{n-2})e^{i(n+1)t}] =$$

$$= O\left(\sum_{k=0}^{n}(n-k)^2|\Delta^3 p_k|+\sum_{k=0}^{n}(n-k)|\Delta^2 p_k|+\sum_{k=0}^{n}|\Delta p_k|+|p_{n-1}|+|p_{n-2}|\right) = o(P_n).$$

Lemma 10. *If there exists an* $l \neq 0$ *such that*

$$\frac{np_n}{P_n} - l = O\left(\frac{1}{n}\right) \quad (n \to \infty)$$

and (N, p) *is regular, then*

$$\text{(i)} \lim_{n\to\infty} |P_n| = \infty, \quad \text{(ii)} \sum_{k=1}^{n} \frac{|P_k|}{k} = O(P_n) \quad \textit{and} \quad \text{(iii)} \sum_{k=1}^{n} k\,|\Delta p_k| = O(P_n).$$

PROOF. (i) Given any $\varepsilon > 0$, there exists an n_0 such that for all $n > n_0$

$$|l| - \varepsilon < \frac{n\,|p_n|}{|P_n|} < |l| + \varepsilon.$$

Since (N, p) is regular, $\sum_1^n |p_k| < A\,|P_n|$, where $A > 0$ is a constant, so that $|P_n| > A^{-1}|p_0| \neq 0$ for all n. Hence for $n > n_0$

$$A^{-1}|p_0|\,\frac{|l| - \varepsilon}{n} < |p_n|$$

and

$$\lim_{n\to\infty} \sum_1^n |p_k| = \infty.$$

Since $\sum_1^n |p_k| < A\,|P_n|$ for all n, $\lim |P_n| = \infty$.

(ii) Since (i) holds and (N, p) is regular, we also have

$$\sum_{k=1}^{n} \frac{|P_k|}{k} \leqq \sum_{k=1}^{n_0} \frac{|P_k|}{k} + \sum_{n_0+1}^{n} \frac{1}{|l| - \varepsilon}\,\frac{k\,|p_k|}{k} \leqq A\,|P_n|.$$

(iii) We have

$$p_n = P_n\left(\frac{l}{n} + O\left(\frac{1}{n^2}\right)\right)$$

so that

$$P_{n-1} = P_n - p_n = P_n\left(1 - \frac{l}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Hence

$$|\Delta p_n| = |p_n - p_{n-1}| = \left|P_n\left(\frac{l}{n} + O\left(\frac{1}{n^2}\right)\right) - P_{n-1}\left(\frac{l}{n-1} + O\left(\frac{1}{n^2}\right)\right)\right| =$$

$$= \left|P_n\left(\frac{l}{n} + O\left(\frac{1}{n^2}\right)\right) - P_n\left(1 - \frac{l}{n} + O\left(\frac{1}{n^2}\right)\right)\left(\frac{l}{n-1} + O\left(\frac{1}{n^2}\right)\right)\right| = O\left(\frac{|P_n|}{n^2}\right)$$

and

$$\sum_{k=1}^{n} k\,|\Delta p_k| = O\left(\sum_{k=1}^{n} \frac{|P_k|}{k}\right) = O(P_n).$$

4. PROOF *of Theorem 1.* We have

$$\frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, kB_k(x)-\frac{D(x)}{\pi}=$$

$$=\int_0^{\pi}\psi(t)\cdot\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\,dt-\frac{(-1)^nD(x)}{\pi P_n}\sum_{k=0}^{n}p_k(-1)^k=$$

$$=\int_0^{\pi}\psi(t)\cdot\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\,dt+o(1)$$

by virtue of lemmas 5, 2 and 3.

Put

$$\int_0^{\pi}\psi(t)\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\,dt=\Big(\int_0^{1/n}+\int_{1/n}^{\delta}+\int_{\delta}^{\pi}\Big)\psi(t)\cdot\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\,dt$$

$$=L_1+L_2+L_3,\quad\text{(say)}.$$

Now

$$L_1=O\Big(\int_0^{1/n}n\cdot|\psi(t)|\,dt\Big)=O\left(n\cdot\Psi\left(\frac{1}{n}\right)\right)=o(1),$$

by virtue of regularity conditions.

And

$$L_3=O\left(\int_{\delta}^{\pi}|\psi(t)|\cdot\left|\frac{1}{P_n}\sum_{k=0}^{n}(n-k)\,p_ke^{ikt}\right|dt\right)=o\Big(\int_{\delta}^{\pi}|\psi(t)|\,dt\Big)=o(1),$$

by virtue of lemmas 5, 2 and 6.

In order to show that $L_2=o(1)$, we make use of an estimate for the kernel in the interval (n^{-1},δ), given by ASTRACHAN [1]. We have

$$\left|\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\right|\leqq A\sum_{j=1}^{\tau}M_{nj}(t)$$

where, setting $\tau=[t^{-1}]$,

$$M_{n1}(t)=\frac{nR(t^{-1})}{R(n)},\quad M_{n2}(t)=\frac{nr(t^{-1})}{tR(n)},$$

$$M_{n3}(t)=\frac{(n-\tau)|\Delta p_\tau|}{t^2R(n)},\quad M_{n4}(t)=\frac{r(t^{-1}-1)}{t^2R(n)},$$

$$M_{n5}(t)=\frac{r(n-1)}{t^2R(n)},\quad M_{n6}(t)=\frac{W(n)-W(t^{-1})}{t^2R(n)}$$

$$M_{n7}(t)=\frac{V(n-1)-V(t^{-1}-1)}{t^2R(n)}$$

and

$$r_n = |p_n|, \quad r(u) = r_{[u]}, \quad R_n = \sum_0^n r_k, \quad R(u) = R_{[u]},$$

$$V_0 = 0, \quad V_n = \sum_1^n |\Delta p_k|, \quad V(u) = V_{[u]},$$

$$W_n = \sum_{k=0}^n (n-k) |\Delta^2 p_k|, \quad W(u) = W_{[u]}.$$

Now

$$L_2 = \int_{1/n}^{\delta} \psi(t) \sum_{k=0}^n k p_{n-k} \sin kt \, dt = O\Big(\int_{1/n}^{\delta} |\psi(t)| \sum_{j=1}^n M_{nj}(t) \, dt \Big).$$

For $j=1$,

$$\int_{1/n}^{\delta} |\psi(t)| \frac{nR(t^{-1})}{R(n)} dt = \left[\Psi(t) \cdot \frac{nR(t^{-1})}{R(n)} \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} \Psi(t) \cdot \frac{n}{R(n)} dR(t^{-1}) =$$

$$= O\left(\frac{n}{|P_n|}\right) + o(1) + o\left(\frac{n}{R(n)} \int_1^n \frac{1}{s} |dR(s)|\right) = o(1) + o\left(\frac{n}{|P_n|} \sum_{k=1}^n \frac{|p_k|}{k}\right) =$$

$$= o(1) + o\left(\frac{n}{|P_n|} \sum_{k=1}^n \frac{|P_k|}{k^2}\right) = o(1)$$

by virtue of lemma 4 and the hypothesis.

For $j=2$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{nr(t^{-1})}{tR(n)} dt$$

$$= \left[\Psi(t) \cdot \frac{nr(t^{-1})}{tR(n)} \right]_{1/n}^{\delta} - \int_{1/n}^{\delta} \Psi(t) \cdot \frac{n \, dr(t^{-1})}{tR(n)} + \int_{1/n}^{\delta} \Psi(t) \cdot \frac{nr(t^{-1})}{R(n)} \cdot \frac{dt}{t^2} =$$

$$= O\left(\frac{n}{R(n)}\right) + o\left(\frac{nr(n)}{R(n)}\right) + o\left(\frac{n}{R(n)} \int_1^n |dr(s)|\right) + o\left(\frac{n}{R(n)} \int_1^n \frac{r(s)}{s} ds\right) =$$

$$= O\left(\frac{n}{|P_n|}\right) + o\left(\frac{n|p_n|}{|P_n|}\right) + o\left(\frac{n}{|P_n|} \sum_{k=1}^n |\Delta p_k|\right) + o\left(\frac{n}{|P_n|} \sum_{k=1}^n \frac{|p_k|}{k}\right) = o(1),$$

by virtue of lemmas 4, 1 and 5.

For $j=3$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{(n-\tau)|\Delta p_\tau)}{t^2 R(n)}\, dt =$$

$$= \left[\Psi(t) \cdot \frac{(n-\tau)|\Delta p_\tau|}{t^2 R(n)}\right]_{1/n}^{\delta} - \int_{1/n}^{\delta} \Psi(t) \cdot \frac{(n-\tau)\, d|\Delta p_\tau|}{t^2 R(n)} +$$

$$+ \int_{1/n}^{\delta} \Psi(t) \cdot \frac{|\Delta p_\tau|\, d\tau}{t^2 R(n)} + 2 \int_{1/n}^{\delta} \Psi(t) \frac{(n-\tau)|\Delta p_\tau|}{t^3 R(n)}\, dt =$$

$$= O\left(\frac{n}{R(n)}\right) + o\left(\frac{1}{R(n)} \int_1^n s(n-[s]) \big|d|\Delta p_{[s]}|\big|\right) +$$

$$+ o\left(\frac{1}{R(n)} \int_1^n |\Delta p_{[s]}|\, d[s]\right) + o\left(\frac{1}{R(n)} \int_1^n (n-[s])\, |\Delta p_{[s]}|\, ds\right) =$$

$$= O\left(\frac{n}{|P_n|}\right) + o\left(\frac{1}{|P_n|} \sum_{k=1}^{n} k(n-k)|\Delta^2 p_k|\right) +$$

$$+ o\left(\frac{1}{|P_n|} \sum_{k=1}^{n} |\Delta p_k|\right) + o\left(\frac{1}{|P_n|} \sum_{k=1}^{n} (n-k)\, |\Delta p_k|\right) = o(1),$$

by virtue of lemmas 4 and 5.
For $j=4$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{r(t^{-1}-1)}{t^2 R(n)}\, dt =$$

$$= \left[\Psi(t) \cdot \frac{r(t^{-1}-1)}{t^2 R(n)}\right]_{1/n}^{\delta} - \int_{1/n}^{\delta} \Psi(t) \cdot \frac{dr(t^{-1}-1)}{t^2 R(n)} + 2 \int_{1/n}^{\delta} \Psi(t) \cdot \frac{r(t^{-1}-1)}{t^3 R(n)}\, dt =$$

$$= O\left(\frac{1}{R(n)}\right) + o\left(\frac{nr(n)}{R(n)}\right) + o\left(\frac{1}{R(n)} \int_1^n s\,|dr(s-1)|\right) + o\left(\frac{1}{R(n)} \int_1^n r(s-1)\, ds\right) =$$

$$= O\left(\frac{1}{|P_n|}\right) + o\left(\frac{n\,|p_n|}{|P_n|}\right) + o\left(\frac{1}{|P_n|} \sum_{k=1}^{n} k\,|\Delta p_k|\right) + o\left(\frac{1}{|P_n|} \sum_{k=1}^{n} |p_k|\right) = o(1),$$

by virtue of lemmas 5, 4 and 1.
For $j=5$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{r(n-1)}{t^2 R(n)}\, dt = \left[\Psi(t) \cdot \frac{r(n-1)}{t^2 R(n)}\right]_{1/n}^{\delta} + 2 \int_{1/n}^{\delta} \Psi(t) \cdot \frac{r(n-1)}{t^2 R(n)}\, dt =$$

$$= O\left(\frac{r(n)}{R(n)}\right) + o\left(\frac{nr(n)}{R(n)}\right) = O\left(\frac{|p_n|}{|P_n|}\right) + o\left(\frac{n\,|p_n|}{|P_n|}\right) = o(1),$$

by virtue of lemmas 5 and 1.

For $j=6$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{W(n)-W(t^{-1})}{t^2R(n)}\,dt =$$

$$= \left[\Psi(t)\cdot\frac{W(n)-W(t^{-1})}{t^2R(n)}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta}\Psi(t)\cdot\frac{dW(t^{-1})}{t^2R(n)} + 2\int_{1/n}^{\delta}\Psi(t)\cdot\frac{W(n)-W(t^{-1})}{t^3R(n)}\,dt =$$

$$= O\left(\frac{W(n)}{R(n)}\right) + o\left(\frac{1}{R(n)}\int_1^n s\,dW(s)\right) + o\left(\frac{1}{R(n)}\int_1^n [W(n)-W(s)]\,ds\right) =$$

$$= O\left(\frac{1}{|P_n|}\sum_{k=1}^{n}(n-k)|\Delta^2 p_k|\right) + o\left(\frac{1}{|P_n|}\sum_{k=0}^{n}k(n-k)|\Delta^2 p_k|\right) = o(1),$$

by virtue of lemma 5.
For $j=7$,

$$\int_{1/n}^{\delta} |\psi(t)| \cdot \frac{V(n-1)-V(t^{-1}-1)}{t^2R(n)}\,dt =$$

$$= \left[\Psi(t)\cdot\frac{V(n-1)-V(t^{-1}-1)}{t^2R(n)}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta}\Psi(t)\cdot\frac{dV(t^{-1}-1)}{t^2R(n)} +$$

$$+2\int_{1/n}^{\delta}\Psi(t)\cdot\frac{V(n-1)-V(t^{-1}-1)}{t^3R(n)}\,dt =$$

$$= O\left(\frac{V(n)}{R(n)}\right) + o\left(\frac{1}{R(n)}\int_1^n s\,|dV(s-1)|\right) + o\left(\frac{1}{R(n)}\int_1^n [V(n-1)-V(s-1)]\,ds\right) =$$

$$= O\left(\frac{1}{|P_n|}\sum_{k=1}^{n}|\Delta p_k|\right) + o\left(\frac{1}{|P_n|}\sum_{k=1}^{n}k|\Delta p_k|\right)$$

by virtue of lemma 5.

Combining these results we prove that $L_2=o(1)$. This completes the proof of the theorem.

5. Proof of Theorem 2.

We have, by virtue of lemmas 7, 2 and 3,

$$\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\cdot kB_k(x) - \frac{D(x)}{\pi} =$$

$$= \int_0^{\pi}\psi(t)\cdot\frac{1}{\pi P_n}\sum_{k=0}^{n}kp_{n-k}\sin kt\,dt + o(1) =$$

$$= -\int_0^{\pi}\psi_1(t)\cdot\frac{1}{\pi P_n}\sum_{k=0}^{n}k^2p_{n-k}\cos kt\,dt + o(1).$$

It suffices to show that the last integral is $o(1)$ $(n\to\infty)$. This integral can be written as the sum of three integrals L_1, L_2, L_3 over the intervals (o, n^{-1}) (n^{-1}, δ) and (δ, π) respectively.

Since $P_n^{-1}\sum_{k=0}^{n} k^2 p_{n-k}\cos kt$ is $O(n^2)$ for $t\in(0, n^{-1})$ and since, by virtue of lemma 9, it is $o(1)$ for $t\in(\delta,\pi)$, it follows that $L_1=o(1)$ and $L_3=o(1)$ $(n\to\infty)$.

Now

$$L_2 = O\left(\int_{1/n}^{\delta} |\psi_1(t)|\cdot\left|\frac{1}{\pi P_n}\sum_{k=0}^{n}(n-k)^2 p_k\cos(n-k)t\right| dt\right) =$$

$$= o\left(\int_{1/n}^{\delta}\left|\frac{t}{P_n}\sum_{k=0}^{n}(n-k)^2 p_k\cos(n-k)t\right| dt\right) = o(1),$$

provided

$$\int_{1/n}^{\delta}\left|\frac{t}{P_n}\sum_{k=0}^{n}(n-k)^2 p_k e^{ikt}\right| dt = O(1).$$

We require a suitable estimate for the integrand in the interval (n^{-1},δ).

Put

$$\sum_{k=0}^{n}(n-k)^2 p_k e^{ikt} = \Sigma_1+\Sigma_2 \quad (t>0),$$

where k ranges over the integers $\leqq\tau=[t^{-1}]$ in Σ_1 and over the integers $>\tau$ but $\leqq n$ in Σ_2. It is clear that

$$|\Sigma_1| \leqq \sum_{k=0}^{\tau}|p_k|\cdot(n-k)^2 \leqq n^2\sum_{k=0}^{\tau}|p_k| = n^2R(t^{-1}).$$

Making three successive Abel's transformation, we see that

$$|\Sigma_2| \leqq A\big[n^2t^{-1}|p_\tau|+n^2t^{-2}|\Delta p_\tau|+nt^{-2}|p_{\tau-1}|+t^{-2}|p_{n-1}|+(n-\tau)^2t^{-3}|\Delta^2 p_\tau|+$$
$$+nt^{-3}|\Delta p_{\tau-1}|+t^{-3}|p_{\tau-2}|+t^{-3}|\Delta p_{n-1}|+t^{-3}|p_{n-2}|+$$
$$+t^{-3}\sum_{\tau+1}^{n}(n-k)^2|\Delta^3 p_k|+t^{-3}\sum_{\tau+1}^{n}(n-k)|\Delta^2 p_k|+t^{-3}\sum_{\tau+1}^{n}|\Delta p_{k-2}|\big].$$

Hence

$$\left|\frac{t}{P_n}\sum_{k=0}^{n}(n-k)^2 p_k e^{ikt}\right| \leqq$$

$$\leqq A\left[\frac{n^2tR(t^{-1})}{R(n)}+\frac{n^2r(t^{-1})}{R(n)}+\frac{n^2t^{-1}|\Delta p_\tau|}{R(n)}+\frac{nt^{-1}r(t^{-1}-1)}{R(n)}+\frac{t^{-1}r(n-1)}{R(n)}+\right.$$
$$+\frac{(n-\tau)^2t^{-2}|\Delta^2 p_\tau|}{R(n)}+\frac{nt^{-2}|\Delta p_{\tau-1}|}{R(n)}+\frac{t^{-2}r(t^{-1}-2)}{R(n)}+\frac{t^{-2}|\Delta p_{n-1}|}{R(n)}+\frac{t^{-2}r(n-2)}{R(n)}+$$
$$\left.+\frac{t^{-2}[\overline{W}(n)-\overline{W}(t^{-1})]}{R(n)}+\frac{t^{-2}[W(n-1)-W(t^{-1}-1)]}{R(n)}+\frac{t^{-2}[V(n-2)-V(t^{-1}-2)]}{R(n)}\right] \leqq$$
$$\leqq A\sum_{j=1}^{13}M_{nj}(t),$$

where $\overline{W}_n=\sum_{k=0}^{n}(n-k)^2|\Delta^3 p_k|$, $\overline{W}(u)\equiv\overline{W}_{[u]}$.

As in the proof of theorem 1, we show that

$$\int_{1/n}^{\delta} M_{nj}(t)\,dt = O(1),$$

by virtue of the hypotheses and lemmas 1, 5 and 7.

Combining the above results we find that $L_2=o(1)$ and this completes the proof of theorem 2.

6. PROOF OF THEOREM 3.

Since

$$\tilde{S}_n(x) = \frac{1}{2\pi}\int_0^{\pi} \{f(x+t)-f(x-t)\}\,\frac{\cos\frac{1}{2}t-\cos\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt$$

and

$$N_n^p\{\tilde{S}(x)\} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\tilde{S}_k(x) =$$

$$= \int_0^{\pi} \psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_{n-k}\,\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt+\frac{D(x)}{P_n}\sum_{k=0}^{n} P_{n-k}\,\frac{1-(-1)^k}{k},$$

we have

$$N_m^p\{\tilde{S}(x)\}-N_n^p\{\tilde{S}(x)\} = \int_0^{\pi} \psi(t)\cdot\frac{1}{2\pi P_m}\sum_{k=0}^{m} p_{m-k}\,\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt-$$

$$-\int_0^{\pi} \psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_{n-k}\,\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt+\frac{D(x)}{\pi P_m}\sum_{k=0}^{m} P_{m-k}\cdot\frac{1-(-1)^k}{k}-$$

$$-\frac{D(x)}{\pi P_n}\sum_{k=0}^{n} P_{n-k}\cdot\frac{1-(-1)^k}{k} =$$

$$= I(x)+\frac{D(x)}{\pi P_m}\sum_{k=0}^{m} P_{m-k}\,\frac{1-(-1)^k}{k}-\frac{D(x)}{\pi P_n}\sum_{k=0}^{n} P_{n-k}\,\frac{1-(-1)^k}{k}.$$

Now

$$I(x)=\int_{1/m}^{1/n}\psi(t)\cdot\frac{1}{2\pi}\cot\frac{1}{2}t\,dt-\int_{1/m}^{\pi}\psi(t)\cdot\frac{1}{2\pi P_m}\sum_{k=0}^{m}p_{m-k}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt+$$

$$+\int_{1/n}^{\pi}\psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt+$$

$$+\int_{0}^{1/m}\psi(t)\cdot\frac{1}{2\pi P_m}\sum_{k=0}^{m}p_{m-k}\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt-$$

$$-\int_{0}^{1/n}\psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt.$$

It is easily seen that, in view of lemma 10,

$$\int_{0}^{1/n}\psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos\frac{1}{2}t-\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt$$

and

$$\int_{1/n}^{\pi}\psi(t)\cdot\frac{1}{2\pi P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,dt$$

are $o(1)$ as $n\to\infty$ (cf. HILLE and TAMARKIN [8]). The same is true for these expressions with n replaced by m. Thus we get

$$I(x)=\int_{1/m}^{1/n}\psi(t)\cdot\frac{1}{2\pi}\cot\frac{1}{2}t\,dt+o(1)=O\left(\int_{1/m}^{1/n}|\psi(t)|\frac{dt}{t}\right)+o(1)=$$

$$=O\left(m\int_{1/m}^{\xi}|\psi(t)|\,dt\right)+o(1)=O\left(m\int_{0}^{\xi}|\psi(t)|\,dt\right)+o(1)=$$

$$=o(m\cdot\xi)+o(1)=o\left(\frac{m}{n}\right)+o(1)=o(1),\quad\text{where}\quad\frac{1}{m}\leqq\xi\leqq\frac{1}{n}.$$

It remains to show that

$$\lim_{n\to\infty} \frac{D(x)}{\pi}\left[\frac{1}{P_m}\sum_{k=1}^{m} P_{m-k}\frac{1-(-1)^k}{k} - \frac{1}{P_n}\sum_{k=1}^{n} P_{n-k}\cdot\frac{1-(-1)^k}{k}\right] = \frac{D(x)}{\pi}\log d.$$

We have

$$\frac{1}{P_m}\sum_{k=1}^{m} P_{m-k}\cdot\frac{1}{k} - \frac{1}{P_n}\sum_{k=1}^{n} P_{n-k}\cdot\frac{1}{k} = \sum_{\nu=n+1}^{m}\left\{\sum_{k=1}^{\nu}\left(\frac{P_{\nu-k}}{P_\nu} - \frac{P_{\nu-k-1}}{P_{\nu-1}}\right)\cdot\frac{1}{k}\right\} =$$

$$= \sum_{\nu=n+1}^{m}\left\{\sum_{k=1}^{\nu-1}\left(\frac{p_{\nu-k}}{P_\nu} - \frac{P_{\nu-k-1}}{P_{\nu-1}}\cdot\frac{p_\nu}{P_\nu}\right)\cdot\frac{1}{k}\right\} + o(1) =$$

$$= \sum_{\nu=n+1}^{m}\frac{1}{\nu P_\nu}\sum_{k=1}^{\nu-1} p_{\nu-k} + \sum_{\nu=n+1}^{m}\frac{1}{\nu P_\nu}\left\{\sum_{k=1}^{\nu-1}\left(\frac{kp_k}{P_{k-1}} - \frac{\nu p_\nu}{P_{\nu-1}}\right)\frac{P_{k-1}}{\nu-k}\right\} + o(1) =$$

$$= \log\frac{m}{n} + o(1) = \log d + o(1),$$

since the hypothesis implies that

$$P_n = P_{n-1} + p_n = P_{n-1} + P_n\left(\frac{l}{n} + \frac{A}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$

or that

$$P_n\left(1 - \frac{l}{n} - \frac{A}{n^2} + o\left(\frac{1}{n^2}\right)\right) = P_{n-1}$$

and

$$\frac{np_n}{P_{n-1}} = \frac{np_n}{P_n}\cdot\frac{P_n}{P_{n-1}} = \left(l + \frac{A}{n} + o\left(\frac{1}{n}\right)\right)\left(1 + \frac{l}{n} + O\left(\frac{1}{n^2}\right)\right) =$$

$$= l + \frac{A+l^2}{n} + o\left(\frac{1}{n}\right) = l + \frac{A'}{n} + o\left(\frac{1}{n}\right)$$

so that given any $\varepsilon > 0$ there exists an n_0 such that

$$\left|\sum_{k=1}^{\nu-1}\left(\frac{kp_k}{P_{k-1}} - \frac{\nu p_\nu}{P_{\nu-1}}\right)\frac{P_{k-1}}{\nu-k}\right| \leqq \frac{1}{\nu-n_0}\sum_{k=1}^{n_0}\left|\frac{kp_k}{P_{k-1}} - \frac{\nu p_\nu}{P_{\nu-1}}\right| |P_{k-1}| + A'\varepsilon\sum_{n_0+1}^{\nu-1}\frac{|P_k|}{k}.$$

Since $\lim\limits_{n\to\infty} P_n^{-1}\sum\limits_{1}^{n} P_{n-k}\dfrac{(-1)^k}{k}$ exists, the assertion holds and the proof of the theorem is thus completed.

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