A universal construction of universal coefficient sequences

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It is shown in this paper[1]) that each R-additive exact connected sequence of functors on R-modules which preserves direct sums gives rise in a natural and unique way to a universal coefficient sequence. One obtains from this most universal coefficient sequences arising from homological algebra and algebraic topology. This refines some results of DOLD [1] while avoiding the unnecessary use of spectral sequences.

1. *The Construction.*

We denote by R a principal ideal domain and by $\mathfrak{m}$ the category of all R-modules. $\mathfrak{m}_0$ denotes throughout a full subcategory of $\mathfrak{m}$ which contains $\{0\}$ and R and which contains enough free modules. By the latter statement is meant that for each G in $\mathfrak{m}_0$ there is an exact sequence $0 \to K \to F \to G \to 0$ in $\mathfrak{m}_0$ with F free. Since R is a principal ideal domain, K must then also be free.

The notation Hom, $\otimes$, and Tor will always mean Hom_R, $\otimes_R$, and Tor_R, respectively. We usually denote $\mathrm{Tor}_1(A, B)$ by $A * B$.

Definition. Let $F\colon \mathfrak{m}_0 \to \mathfrak{m}$ be a covariant functor. A *UC-function for F* is a natural transformation

$$\tau : F(R) \otimes - \to F$$

which extends the canonical isomorphism $F(R) \otimes R \to F(R)$.

Lemma 1. *Let τ and τ' be UC-functions for, respectively, the covariant functors $F, F'\colon \mathfrak{m}_0 \to \mathfrak{m}$. If $s\colon F \to F'$ is a natural transformation, then the diagram*

$$\begin{array}{ccc} F(R) \otimes G & \xrightarrow{\ \tau\ } & F(G) \\ \big\downarrow{\scriptstyle s_R \otimes 1} & & \big\downarrow{\scriptstyle s_G} \\ F'(R) \otimes G & \xrightarrow{\ \tau'\ } & F'(G) \end{array}$$

commutes for each G in $\mathfrak{m}_0$.

PROOF. The diagram commutes for $G = R$ since s_R is a homomorphism. The commutativity for arbitrary G then follows by observing that each generator $h \otimes g$ of $F(R) \otimes G$ is in the image of some homomorphism $F(R) \otimes R \xrightarrow{1 \otimes \alpha} F(R) \otimes G$.

A functor $F\colon \mathfrak{m}_0 \to \mathfrak{m}$ is called *R-additive* if, whenever $\alpha, \beta \in \mathrm{Hom}(A, B)$, A, B in $\mathfrak{m}_0$, and $r \in R$, one has $F(\alpha + \beta) = F(\alpha) + F(\beta)$ and $F(r\alpha) = rF(\alpha)$.

Theorem A. *If $F\colon \mathfrak{m}_0 \to \mathfrak{m}$ is an R-additive covariant functor, then there exists a unique* UC-*function for F.*

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PROOF. Since F is R-additive, it defines an R-module homomorphism $\mathrm{Hom}\,(R, G) \to \mathrm{Hom}\,(F(R), F(G))$ which when composed with the canonical isomorphism $G \to \mathrm{Hom}\,(R, G)$ yields a homomorphism

$$\varphi : G \to \mathrm{Hom}\,(F(R), F(G)).$$

This homomorphism induces through the correspondence $h \otimes g \mapsto \varphi(g)h$ a homomorphism

$$\tau : F(R) \otimes G \to F(G).$$

From the defining properties of a functor it follows that τ is a UC-functor for F. The uniqueness of τ follows from Lemma 1 by taking s to be the identity natural transformation from F to itself.

We denote by $\mathfrak{m}^\infty$ the category of graded R-modules.

Definition. (F, δ) is a *connected sequence of (covariant) functors* on $\mathfrak{m}_0$ if $F: \mathfrak{m}_0 \to \mathfrak{m}^\infty$ is a covariant functor and for each integer p and each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{m}_0$, $\delta: F^p(C) \to F^{p+1}(A)$ is a homomorphism such that if

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

is a commutative diagram in $\mathfrak{m}_0$ with exact rows, then the diagram

$$\begin{array}{ccc} F^p(C) & \xrightarrow{\ \delta\ } & F^{p+1}(A) \\ \downarrow{\scriptstyle F(\gamma)} & & \downarrow{\scriptstyle F(\alpha)} \\ F^p(C') & \xrightarrow{\ \delta\ } & F^{p+1}(A') \end{array} \quad \text{commutes.}$$

(F, δ) is called *exact* if the sequence

$$\ldots \to F^{p-1}(C) \xrightarrow{\ \delta\ } F^p(A) \xrightarrow{F^p(\lambda)} F^p(B) \xrightarrow{F^p(\varrho)} F^p(C) \xrightarrow{\ \delta\ } F^{p+1}(A) \to \ldots$$

is exact for each exact sequence

$$0 \to A \xrightarrow{\ \lambda\ } B \xrightarrow{\ \varrho\ } C \to 0 \text{ in } \mathfrak{m}_0.$$

For any connected sequence of functors (F, δ), the preceding long exact sequence can easily be shown to be of order two (i.e., the composite of two successive homomorphisms is zero), whenever $F^p(0)=0$ (e.g., F^p is additive) for each p.

If G is in $\mathfrak{m}$ and $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathfrak{m}$, then we let

$$E_* : G * C \to G \otimes A$$

denote the connecting homomorphism in the canonical long exact sequence for Tor.

Definition. Let (F, δ) be a connected sequence of functors on $\mathfrak{m}_0$. (τ, σ) is a *UC-pair for* (F, δ) if

(i) τ^p is a UC-function for F^p for each integer p, and

(ii) σ^p is, for each integer p, a natural transformation $F^p \to F^{p+1}(R) *$ — such that if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathfrak{m}_0$, then the diagram

$$\begin{array}{ccc} F^p(C) & \xrightarrow{\ \sigma^p\ } & F^{p+1}(R) * C \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle E_*} \\ F^{p+1}(A) & \xrightarrow{\ \tau^{p+1}\ } & F^{p+1}(R) \otimes A \end{array}$$

commutes.

We show below that the defining properties of a *UC*-pair serve to characterize the usual universal coefficient sequences. We first need a Lemma.

Lemma 2. *If τ is a UC-function for the functor $F\colon \mathfrak{m}_0 \to \mathfrak{m}$ and if F preserves direct sums, then τ_F is an isomorphism for each free module F in $\mathfrak{m}_0$.*

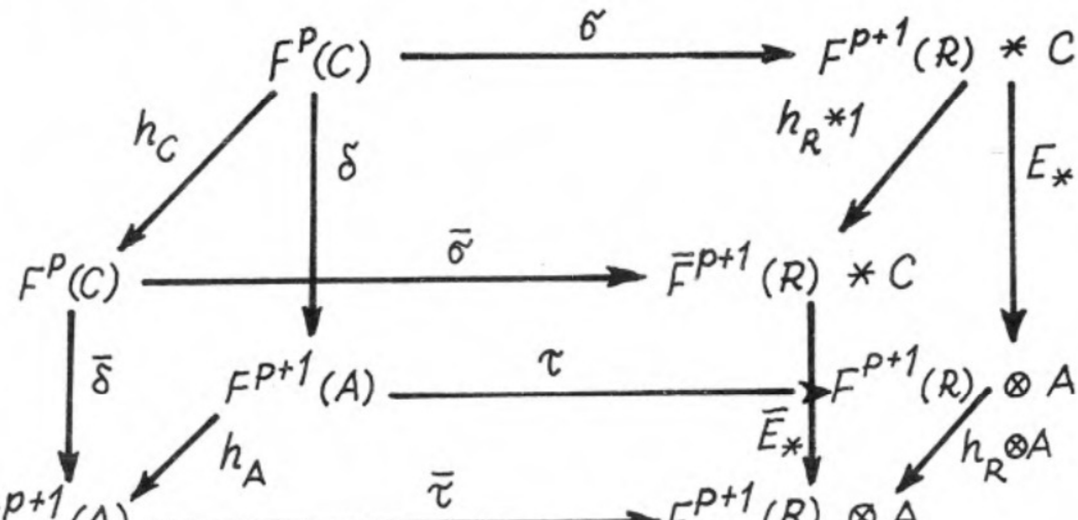
Proof. For a free module G in $\mathfrak{m}_0$, the canonical isomorphism extends to an isomorphism $F(R)\otimes G \to F(G)$ since G decomposes into a direct sum of copies of R and both F and $\otimes$ preserve direct sums.

Theorem B. *Suppose that (τ, σ) and $(\bar\tau, \bar\sigma)$ are UC-pairs for the connected sequences of functors (F, δ) and $(\bar F, \bar\delta)$ on $\mathfrak{m}_0$, respectively, and let $h\colon F \to \bar F$ be a natural transformation. If F preserves direct sums, then the diagram*

$$\begin{array}{ccccc} F^p(R)\otimes G & \xrightarrow{\tau^p} & F^p(G) & \xrightarrow{\sigma^p} & F^{p+1}(R) * G \\ \downarrow{\scriptstyle h_R^p\otimes 1} & & \downarrow{\scriptstyle h_G^p} & & \downarrow{\scriptstyle h_R^{p+1} * 1} \\ \bar F^p(R)\otimes G & \xrightarrow{\bar\tau^p} & \bar F^p(G) & \xrightarrow{\bar\sigma^p} & \bar F^{p+1}(R) * G \end{array}$$

commutes for each G in $\mathfrak{m}_0$ and each p if, and only if, h commutes with δ and $\bar\delta$.

Proof. We first suppose that the diagram commutes. The fact that h commutes with δ and $\bar\delta$ can then be read off the diagram



which arises from an exact sequence $0\to A\to B\to C\to 0$ in $\mathfrak{m}_0$. If, conversely, h commutes with δ and $\bar\delta$ and for given C in $\mathfrak{m}_0$ we choose A and B to be free, then the commutativity of the right rectangle in the diagram of the Theorem (with $G=C$) is contained in the above diagram since in this case $\bar E_*$ and $\bar\tau$ are monomorphisms. The left rectangle commutes from Lemma 1.

Theorem C. *Suppose that (F, δ) is an R-additive* (i.e. F^p is R-additive for each *ps connected sequence of functors on $\mathfrak{m}_0$. If F preserves direct sums, then there exists a unique UC-pair for (F, δ).*

PROOF. The uniqueness of the UC-pair follows from Theorem B by taking $F=\bar{F}$ and $h=1_F$. The existence of τ follows from Theorem A.

For G in $\mathfrak{m}_0$, there is an exact sequence $0\to F_1\to F_0\to G\to 0$ in $\mathfrak{m}_0$ with F_0, F_1 free. The diagram

$$\begin{array}{ccccc} 0 \to F^{p+1}(R)*G & \xrightarrow{E_*} & F^{p+1}(R)\otimes F_1 & \longrightarrow & F^{p+1}(R)\otimes F_0 \\ \uparrow{\scriptstyle \sigma^p} & & \downarrow{\scriptstyle \tau_{F_1}} & & \downarrow{\scriptstyle \tau_{F_0}} \\ F^p(G) & \xrightarrow{\delta} & F^{p+1}(F_1) & \longrightarrow & F^{p+1}(F_0) \end{array}$$

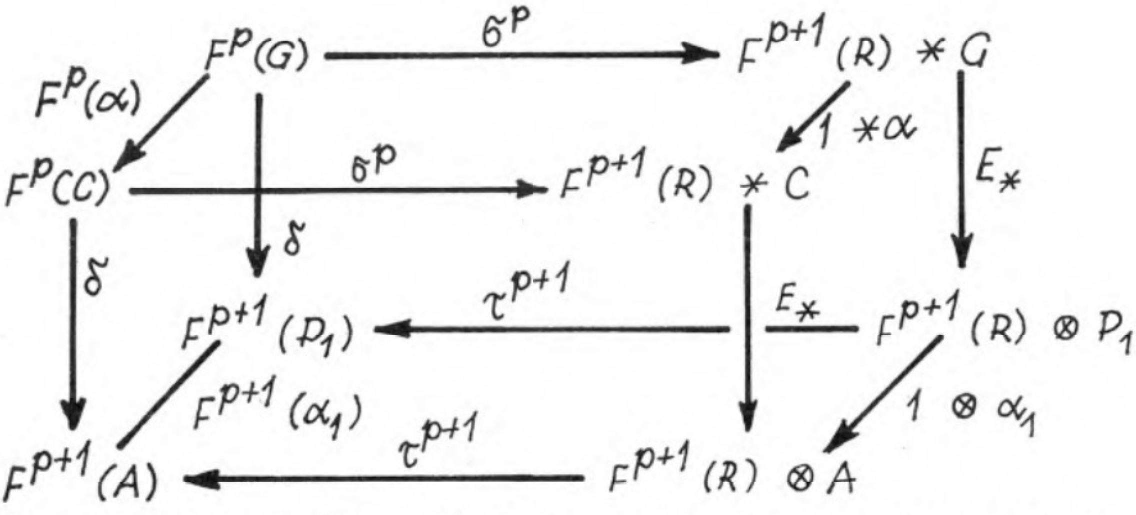
is commutative with τ_{F_0} and τ_{F_1} isomorphisms (Lemma 2). The first row is exact (F_0 is free) and the second row is of order two. There then exists a unique homomorphism σ^p completing the diagram so that it is commutative.

Next let $0\to A\to B\to C\to 0$ be an exact sequence in $\mathfrak{m}_0$ and $\alpha\colon G\to C$ be a homomorphism. The homomorphism α lifts to homomorphisms α_0 and α_1 making the diagram

$$\begin{array}{ccccccccc} 0 & \to & F_1 & \to & F_0 & \to & G & \to & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \to & A & \longrightarrow & B & \longrightarrow & C & \to & 0 \end{array}$$

commutative.

This gives rise to the diagram



in which the commutativity of all sides is clear except the top and front. By taking $G=C$ and $\alpha=1_C$, we obtain the commutativity of the front rectangle (which is the diagram in the definition of a UC-pair). By taking the sequence $0\to A\to B\to C\to 0$ to be the sequence relative to which σ_C^p was defined (A and B free), one then obtains the commutativity of the top rectangle since E_* and τ become monomorphisms. That is, σ^p is a natural transformation.

Theorem D. *Suppose that* (τ, σ) *is a UC-pair for the connected sequence of functors* (F, δ) *on* $\mathfrak{m}_0$. *Then the following are equivalent:*

A. *The sequence*

$$0 \to F^p(R)\otimes G \xrightarrow{\tau} F^p(G) \xrightarrow{\sigma} F^{p+1}(R) * G \to 0$$

is exact for each G *in* $\mathfrak{m}_0$ *and each integer* p.

B. (F, δ) *is exact and* F *preserves direct sums.*

Proof. We first show that B implies A.

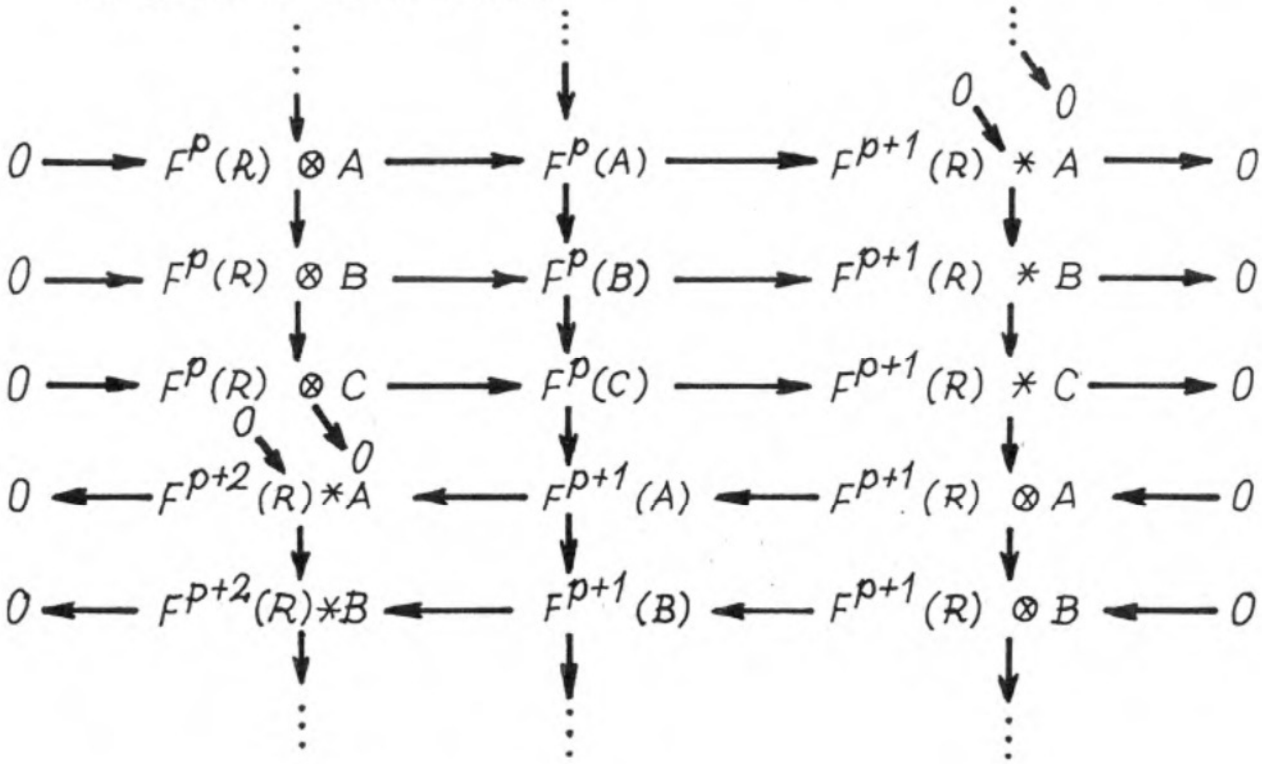
For G in $\mathfrak{m}_0$, there is an exact sequence $0 \to F_1 \to F_0 \to G \to 0$ in $\mathfrak{m}_0$ with F_0, F_1 free. The diagram

$$\begin{array}{ccccccccc}
 & & F^p(R)\otimes F_1 & \xrightarrow[\cong]{\tau} & F^p(F_1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & F^p(R)\otimes F_0 & \xrightarrow[\cong]{\tau} & F^p(F_0) & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^p(R)\otimes G & \xrightarrow{\tau} & F^p(G) & \xrightarrow{\sigma} & F^{p+1}(R) * G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & F^{p+1}(F_1) & \xleftarrow[\cong]{\tau} & F^{p+1}(R)\otimes F_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F^{p+1}(F_0) & \xleftarrow[\cong]{\tau} & F^{p+1}(R)\otimes F_0 & &
\end{array}$$

is commutative with exact columns and with τ_{F_0} and τ_{F_1} isomorphisms. A diagram chase then shows the sequence of part A to be exact.

We turn now to the implication of B from A. That F preserves direct sums is immediate since tensor and torsion products preserve direct sums and τ and σ are natural transformations.

To show that (F, δ) is exact, let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{m}_0$ and consider the following diagram.



The diagram is commutative and has exact rows. The first and third columns consist piecewise of long exact sequences for Tor. In our case, they each have only six terms because R is a principal ideal domain. A diagram chase then yields the exactness of the middle column.

Theorems C and D can be combined to give the following.

Corollary. *If (F, δ) is an R-additive exact connected sequence of functors on $\mathfrak{m}_0$ which preserves direct sums, then there exists for each p a functorial short exact sequence*

$$0 \to F^p(R)\otimes G \to F^p(G) \to F^{p+1}(R) * G \to 0.$$

2. *Examples.*

We give here a few examples of applications of the preceding results.

(i) Let C^* be a fixed cochain complex in $\mathfrak{m}$ and let $\mathfrak{m}_0=\{G\in \mathfrak{m} | C^* * G$ is acyclic$\}$. Then $\mathfrak{m}_0$ contains $\{0\}$ and R and contains enough free modules. Then $H^*(C^*\otimes -)$ is an R-additive exact connected sequence of functors on $\mathfrak{m}_0$ which preserves direct sums, where H^* is the usual homology functor on cochain complexes. One then obtains on $\mathfrak{m}_0$ the universal coefficient sequence [4, p. 236]

$$0 \to H^p(C^*)\otimes G \to H^p(C^*\otimes G) \to H^{p+1}(C^*) * G \to 0.$$

(ii) For a fixed free chain complex C, $H^*(\mathrm{Hom}\,(C, -))$ is an R-additive exact connected sequence of functors. If $H(C)$ is of finite type, then $H^*(\mathrm{Hom}\,(C, -))$ preserves direct sums on $\mathfrak{m}$ (see [4, p. 247]). In this case one obtains the exact sequence [4, p. 246]

$$0 \to H^p(C; R)\otimes G \to H^p(C; G) \to H^{p+1}\,(C; R) * G \to 0,$$

where $H^p(C; G)$ denotes $H^p(\mathrm{Hom}\,(C; G))$.

(iii) For a fixed topological pair (X, A) both the Alexander—Cech and singular cohomology functors $H^*(X, A; -)$ (as well as others) define an R-additive exact connected sequence of functors. It preserves direct sums on $\mathfrak{m}$ in the Alexander—Cech case provided (X, A) is a compact pair and in the singular case provided (X, A) is a locally contractible (e.g. CW-complex) compact pair (A relatively simple proof of this can be based on the main theorem of [3].) One then obtains in these cases the exact sequence [4, p. 336]

$$0 \to H^p(X, A; R)\otimes G \to H^p(X, A; G) \to H^{p+1}(X, A; R) * G \to 0.$$

(iv) For a fixed topological pair (X, A) the singular homology functor $H(X, A; -)$ defines an R-additive exact connected sequence of functors (with $F^p=H_{-p}(X, A; -)$) which preserves direct sums on $\mathfrak{m}$. There is then on $\mathfrak{m}$ the exact sequence

$$0 \to H_p(X, A; R)\otimes G \to H_p(X, A; G) \to H_{p-1}(X, A; R) * G \to 0.$$

3. *Remarks.*

(i) The functoriality in the first variable (as well as the second) of the sequences of the preceding examples also follows from our results. For example, if in example (iii), f: $(X, A)\to(Y, B)$ is a continnous function, then f^* defines a natural transformation from $H^*(Y, B: -)$ to $H^*(X, A; -)$ commuting with the appropriate δ's so that Theorem B applies.

(ii) Each of the sequences in the preceding examples split although it does not follow from our results. Conditions for the splitting of the sequence of the Corollary broad enough to cover the sequences of the examples appear to be unknown. In the case of abelian groups ($R=Z$), it follows from results of HILTON and DELEANU [2] that the sequence of the Corollary splits provided $F^{p+1}(Z)*G$ is a direct sum of cyclic groups. This occurs in particular if the torsion subgroup of either $F^{p+1}(Z)$ or G has finite exponent.

(iii) Since an R-additive functor preserves finite direct sums, the assumption that F preserves direct sums has no force in the preceding results if $\mathfrak{m}_0$ consists of only finitely generated modules. For finitely generated G one obtains, for instance, the sequence of example (iii) without the assumption that (X, A) is a compact pair (resp., locally contractible).

(iv) The familiar universal coefficient sequence

$$0 \to \operatorname{Ext}(H_{p-1}(C),G) \to H^p(\operatorname{Hom}(C, G)) \to \operatorname{Hom}(H_p(C), G) \to 0,$$

where C is a free chain complex, is not contained in our results. We include, for completeness, a sketch of a somewhat conceptual proof of this which stems from DOLD [1, p. 77]. We assume that G has a resolution $0-G\to J_0\to J_1\to 0$ with J_0, J_1 injective. As noted earlier, $H^*(\operatorname{Hom}(C, -))$ is for free C an exact connected sequence of functors so one has the commutative diagram

$$\begin{array}{ccccc}
 & & H^p(\operatorname{Hom}(C, J_1)) & \xrightarrow{\alpha_{J_1}} & \operatorname{Hom}(H_p(C), J_1) \\
 & & \uparrow & & \uparrow \\
0 & & H^p(\operatorname{Hom}(C, J_0)) & \xrightarrow{\alpha_{J_0}} & \operatorname{Hom}(H_p(C), J_0) \\
\uparrow & & \uparrow & & \uparrow \\
\operatorname{Ext}(H_{p-1}(C), G) & \xrightarrow{\beta} & H^p(\operatorname{Hom}(C, G)) & \xrightarrow{\alpha_G} & \operatorname{Hom}(H_p(C), G) \\
\uparrow & & \uparrow & & \uparrow \\
\operatorname{Hom}(H_{p-1}(C), J_1) & \xleftarrow{\alpha_{J_1}} & H^{p-1}(\operatorname{Hom}(C, J_1)) & & 0 \\
\uparrow & & \uparrow & & \\
\operatorname{Hom}(H_{p-1}(C), J_0) & \xleftarrow{\alpha_{J_0}} & H^{p-1}(\operatorname{Hom}(C, J_0)) & &
\end{array}$$

with exact columns ($\operatorname{Ext}(H_{p-1}(C), J_0)=0$ since J_0 is injective), where α denotes the canonically defined homomorphisms. For injective J, $\operatorname{Hom}(-, J)$ is an exact functor so must commute with the homology functor of chain (and cochain) complexes. α_{J_0} and α_{J_1} are therefore isomorphisms so a unique compatible homomorphism β exists giving the sought sequence.

(v) There exists an extension of our results to the case where R is not necessarily a principal ideal domain but the limited utility of the generalization seems not to warrant the inclusion here of the considerably more complex proof.

References

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