Fixed point theorems for some classes of contraction mappings on nonarchimedean probabilistic metric space

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1. K. MENGER [8] initiated the study of probabilistic metric spaces in 1942. In 1958 B. SCHWEIZER and A. SKLAR took up the investigation of these spaces and since then have developed many aspects of their theory.

In [12], [14] are proved some fixed point theorems for contraction mappings defined in [12, Def. 5] or strict contractions [14, Def. 33].

It is our purpose in the present Note to give an extension of some results of V. M. SEGHAL, A. T. BHARUCHA—REID [12] and H. SHERWOOD [14] for nonarchimedean Menger spaces.

2. We first recall [10] that a t-norm is a function $T:[0, 1]\times[0, 1]\to[0, 1]$ which is associative, commutative, nondecreasing in each place and satisfies $T(a, 1)=a$ for each $a\in[0, 1]$. A t-norm T is archimedean ([7], [11]) if in addition to satisfying the above conditions, it is continuous on $[0, 1]\times[0, 1]$ and $T(x, x)<x$ for all $x\in(0, 1)$. A characterization of archimedean t-norms is due to C-H. LING [7] which proved that a t-norm T is archimedean iff it admite the representation

$$T(x, y) = g^{(-1)}\big(g(x)+g(y)\big)$$

where g is continuous and decreasing function from $[0, 1]$ into $[0, \infty]$ with $g(1)=0$ and $g^{(-1)}$ is the preudo-inverse of g.

The continuous, decreasing function g appearing in this characterization is called an additive generator of the archimedean t-norm T.

A nonarchimedean Menger space is an ordered triple $(S, \mathcal{F}, T)$ where $(S, \mathcal{F})$ is a nonarchimedean PM-space, T is a t-norm and the nonarchimedean Menger triangle inequality

$$(IV_m) \qquad F_{pq}(\max\{x,y\}) \geqq T\big(F_{pr}(x), F_{rq}(y)\big)$$

holds for all p, q, r in S and all $x, y\geqq 0$, where $\mathcal{F}(p, q)=F_{pq}$ is a mapping from $S\times S$ into the set of nondecreasing, left-continuous functions F such that $F(0)=0$ and $\lim_{x\to\infty} F(x)=1$.

Definition 2.1. [13] Let $(S, \mathcal{F})$ be a nonarchimedean probabilistic metric space; let g be an additive generator and let k be a number such that $0<k<1$. A mapping $f: S\to S$ is a contraction map on S with respect to g and k if for every $p, q\in S$

$$g\circ F_{f(p)f(q)}(x) \leqq kgF_{pq}\left(\frac{x}{k}\right).$$

Theorem 2.1. *Let $(S, \mathcal{F}, T)$ be a complete nonarchimedean Menger space under the archimedean t-norm T with additive generator g. Let f be a contraction map on S with respect to g and k where $0<k<1$. Then there is a unique point p in S such that $f(p)=p$.*

Proof. Let q be an arbitrary element of S. Define a sequence $\{p_n\}$ inductively via $p_1=f(q)$ and $p_{n+1}=f(p_n)$ for every positive integer n. It follows by induction that for every positive integer n,

$$g\left(F_{p_n p_{n+1}}(x)\right) \leqq k^n g\left(F_{qf(p)}\left(\frac{x}{k^n}\right)\right).$$

Then for $m>n$ and $x>0$ we have

$$F_{p_n p_m}(x) \geqq T\left(F_{p_n p_{n+1}}(x), F_{p_{n+1} p_m}(kx)\right) \geqq$$
$$\geqq T\left(T(F_{p_n p_{n+1}}(x), F_{p_{n+1} p_{n+2}}(kx)), F_{p_{n+2} p_m}(k^2 x)\right) =$$
$$= g^{(-1)}\left[g\left(T(F_{p_n p_{n+1}}(x), F_{p_{n+1} p_{n+2}}(kx))\right) + g\left(F_{p_{n+2} p_m}(k^2 x)\right)\right] =$$
$$= g^{(-1)}\left\{g\left[g^{(-1)}\left(g(F_{p_n p_{n+1}}(x)) + g(F_{p_{n+1} p_{n+2}}(kx))\right)\right] + g\left(F_{p_{n+2} p_m}(k^2 x)\right)\right\} \geqq$$
$$\geqq g^{(-1)}\left\{g\left[g^{(-1)}\left(k^n g\left(F_{qp_1}\left(\frac{x}{k^n}\right)\right)\right) + k^{n+1} g\left(F_{qp_1}\left(\frac{x}{k^n}\right)\right)\right] + k^{n+2} g\left(F_{qp_{m-2}}\left(\frac{x}{k^n}\right)\right)\right\}.$$

A sequence of points $\{p_n\}$ in S is a Cauchy sequence if $F_{p_n p_m} \to H$ (pointwise) as $n, m \to \infty$ where $H(x)=0$ if $x \leqq 0$ and $H(x)=1$ if $x>0$.

We conclude that $\{p_n\}$ is a Cauchy sequence since $g^{(-1)}$ and g are continuous, $k^n \to 0$ as $n \to \infty$, $F_{pq}(x) \to 1$ as $x \to \infty$ and $g^{(-1)}(0)=1$. Since $(S, \mathcal{F}, T)$ is complete there is a point p such that $\{p_n\} \to p$. For every x and for every positive integer n,

$$F_{pf(p)}(x) \geqq T\left(F_{pp_n}(x), F_{p_n f(p)}(x)\right) = g^{(-1)}\left(g(F_{pp_n}(x)) + g(F_{p_n f(p)}(x))\right) \geqq$$
$$\geqq g^{(-1)}\left(g\left(F_{pp_n}(x)\right) + kg\left(F_{p_{n-1} p}\left(\frac{x}{k}\right)\right)\right) \geqq$$
$$\geqq \lim_{n \to \infty} g^{(-1)}\left(g\left(F_{pp_n}(x)\right) + kg\left(F_{p_{n-1} p}\left(\frac{x}{k}\right)\right)\right) = 1$$

i.e., $p=f(p)$ and p is a fixed point of f.

One can prove the uniqueness of the fixed point as in [13] where T is a strict t-norm.

Remarks. A t-norm is strict if T is continuous on $[0, 1]\times[0, 1]$ and strictly increasing in each place on $(0, 1]\times(0, 1]$.

The t-norms $\operatorname{Prod}(x, y)=xy$ and $T_m(x, y)=\operatorname{Max}\{x+y-1, 0\}$ are archimedean. Prod is additively generated by $-\log|_{[0,1]}$; and T_m by the function g_0 defined by $g_0(x)=1-x$ and

$$g_0^{(-1)}(x) = \begin{cases} 1-x, & x \leqq 1 \\ 0, & x > 1. \end{cases}$$

The t-norm Min admits no additive generators [1], [7].

A very natural definition of a contraction map in probabilistic metric spaces was suggested and studied in [12].

Definition 2.2. Let $(S, \mathscr{F})$ be a nonarchimedean probabilistic metric space. A mapping $f\colon S\to S$ is a contraction map on $(S, \mathscr{F})$ if and only if there exists a constant $k\in(0,1)$ such that

$$F_{f(p)f(q)}(x) \geqq F_{pq}\left(\frac{x}{k}\right)$$

for every $p, q\in S$.

Theorem 2.2. *Let $(S, \mathscr{F}, T)$ be a complete nonarchimedean Menger space, where T is a continuous t-norm satisfying $T(x,x)\geqq x$ for each $x\in[0,1]$. If f is any contraction mapping of S into itself then there is a unique $p\in S$ such that $f(p)=p$. Moreover, $f^n(q)\to p$ for each $q\in S$.*

Proof. To prove the existence of the fixed point, consider an arbitrary $q\in S$, and define $p_n=f^n(q)$, $n=1,2,\dots$. We show that the sequence $\{p_n\}$ is fundamental in $(S, \mathscr{F}, T)$. Let ε, λ be positive reals. Then for $m>n$ we have

$$F_{p_n p_m}(\varepsilon) \geqq T\big(F_{p_n p_{n+1}}(\varepsilon), F_{p_{n+1} p_m}(\varepsilon)\big) \geqq T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), F_{p_{n+1}p_m}(k\varepsilon)\right) \geqq$$

$$\geqq T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), F_{p_{n+2}p_m}(k^2\varepsilon)\right)\right) \geqq T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), F_{p_{n+2}p_m}(k^2\varepsilon)\right)$$

since $S_{pq}(\max\{x,y\})\geqq T\big(F_{pr}(x), F_{rq}(y)\big)$ and T is associative.

Using the same argument repeatedly, we obtain

$$F_{p_n p_m}(\varepsilon) \geqq T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), F_{p_{m-1}p_m}(k^{m-n-1}\varepsilon)\right) \geqq T\left(F_{qp_1}\left(\frac{\varepsilon}{k^n}\right), F_{qp_1}\left(\frac{\varepsilon}{k^n}\right)\right) \geqq F_{qp_1}\left(\frac{\varepsilon}{k^n}\right).$$

Therefore, if we choose n_0 such that $F_{qp_1}\left(\dfrac{\varepsilon}{n_0}\right)>1-\lambda$ it follows that $F_{p_n p_m}(\varepsilon)>1-\lambda$ for all $n\geqq n_0$. Hence $\{p_n\}$ is a fundamental sequence. Since $(S, \mathscr{F}, T)$ is a complete nonarchimedean Menger space, there is a $p\in S$ such that $p_n\to p$, i.e., $f^n(q)\to p$.

Let $U_{f(p)}(\varepsilon, \lambda)$ be any neighborhood of $f(p)$. Then $p_n\to p$ implies the existence of an integer n_0 such that $p_n\in U_p(\varepsilon,\lambda)$ for all $n\geqq n_0$. However

$$F_{f(p_n)f(p)}(\varepsilon) \geqq F_{p_n p}\left(\frac{\varepsilon}{k}\right) \geqq F_{p_n p}(\varepsilon) > 1-\lambda$$

for all $n\geqq n_0$. Therefore $f(p_n)\in U_{f(p)}(\varepsilon,\lambda)$ for all $n\geqq n_0$ i.e., $f^n(q)\to f(p)$. Since $(S, \mathscr{F}, T)$ is a Hausdorff space [4] we conclude that $f(p)=p$. This proves the existence point of the theorem.

The proof of the uniqueness part is the same as in [12].

Corollary 2.1. Let (S, d) be a complete nonarchimedean metric space and let $f\colon S\to S$ for which there exists $k\in(0,1)$ such that $d\big(f(p), f(q)\big)\leqq kd(p,q)$ for all $p, q\in S$. Then f has a unique fixed point $p_0\in S$ and $f^n(q)\to p_0$ for each $q\in S$.

PROOF. If $F_{pq}(x)=H(x-d(p,q))$, $p,q\in S$, $x\in R$ then $(S,\mathcal{F},\mathrm{Min})$ is a non-archimedean Menger space which is complete if the metric d is complete. Then since for each $x>0$

$$F_{f(p)f(q)}(kx)=H(kx-d(f(p),f(q)))\geqq H(kx-kd(p,q))=H(x-d(p,q))=F_{pq}(x)$$

it follows that f is a contraction on S into itself and the conclusion follows by Theorem 2.2.

Remark. Since t-norm $\mathrm{Min}(x,y)$ is the strongest possible universal t-norm [10] and $T(x,x)\geqq x$ it follows that in Theorem 2.2, $T(x,y)=\mathrm{Min}(x,y)$. In [14] it is proved that for $T=T_m$ and $T=\mathrm{Prod}$ there exists a contraction map on the Menger space $(S,\mathcal{F},T)$ having no fixed point.

For to prove a similar result for the mappings which are local contractions on a nonarchimedean probabilistic metric space we recall that the family

$$U_p(\varepsilon,\lambda)=\{q\in S: F_{pq}(\varepsilon)>1-\lambda,\ \varepsilon>0,\ \lambda>0,\ p\in S\}$$

is a basis for a Hausdorff nonarchimedean uniformity on nonarchimedean Menger space $(S,\mathcal{F},\mathrm{Min})$ [4].

Definition 2.3. [12] Let $\varepsilon>0$ and $\lambda>0$. A mapping $f\colon S\to S$ is called an (ε,λ)-local contraction if there exists $k\in(0,1)$ such that if $p\in S$ and $q\in U_p(\varepsilon,\lambda)$ then

$$F_{f(p)f(q)}(kx)\geqq F_{pq}(x)\quad\text{for}\quad x>0$$

The nonarchimedean probabilistic metric space $(S,\mathcal{F})$ is called (ε,λ)-chainable if for each $p,q\in S$ there exists a finite sequence $p=p_0,p_1,\dots,p_n=q$ of elements in S such that $p_{i+1}\in U_{p_i}(\varepsilon,\lambda)$, i.e., $F_{p_{i+1}p_i}(\varepsilon)>1-\lambda$ for $i=0,1,\dots,n-1$.

Theorem 2.3. *Let $(S,\mathcal{F},T)$ be a complete nonarchimedean (ε,λ)-chainable Menger space, where $T=\mathrm{Min}$. If $f\colon S\to S$ is an (ε,λ)-local contraction then f has a unique fixed point $p_0\in S$ and $f^n(p)\to p_0$ for each $p\in S$.*

PROOF. We first prove that for each $p\in S$ and for positive real x, there exists a positive integer $n(p,x)$ such that $F_{f^m(p)f^{m+1}(p)}(x)>1-\lambda$ for all $m\geqq n(p,x)$. The reason is similar as in [12].

Let $p=p_0,p_1,\dots,p_m=f(p)$ be a finite sequence such that $F_{p_{i+1}p_i}(\varepsilon)>1-\lambda$, $i=0,1,\dots,n-1$. Since f is (ε,λ)-local contraction it follows that $F_{f(p_{i+1})f(p_i)}(\varepsilon)\geqq \geqq F_{p_{i+1}p_i}\left(\frac{\varepsilon}{k}\right)>1-\lambda$ and that the sequence of elements $f(p_0),f(p_1),\dots,f(p_n)$ is an (ε,λ)-chain for $f(p)$ and $f^2(p)$ and hence by induction $f^r(p),\dots,f^r(p_1),\dots,f^r(p_n)$ is an (ε,λ)-chain for $f^r(p)$ and $f^{r+1}(p)$ for each positive integer r. Therefore for $x>0$, and for each integer $r>0$

$$F_{f^r(p_{i+1})f^r(p_i)}(x)\geqq F_{p_{i+1}p_i}\left(\frac{x}{k^r}\right)$$

and

$$F_{f^r(p_0)f^r(p_n)}(x)\geqq T\left(F_{p_0p_1}\left(\frac{x}{k^r}\right),\ F_{p_1p_n}\left(\frac{x}{k^r}\right)\right)\geqq$$

$$\geqq T\left(F_{p_0p_1}\left(\frac{x}{k^r}\right),\ T\left(F_{p_1p_2}\left(\frac{x}{k^r}\right),\ F_{p_2p_n}\left(\frac{x}{k^r}\right)\right)\right).$$

It follows by induction that

$$F_{f^r(p_0)f^r(p_n)}(x) \geqq T\big(F_{p_0p_1}(x/k^r, T(F_{p_1p_2}(x/k^r)), \dots, T\big(F_{p_{n-2}p_{n-1}}(x/k^r), F_{p_{n-1}p_n}(x/k^r)\big)\dots\big).$$

Since n is a fixed finite integer, there exists an integer $m_i > 0$ such that $F_{p_i p_{i+1}}(x/k^r) > 1-\lambda$ for each $r > m_i$, $i = 0, 1, \dots, n-1$. Let $n(p, x) = \max\{m_0, m_1, \dots, m_{n-1}\}$. Then $F_{f^r(p)f^{r+1}(p)}(x) > 1-\lambda$ for all $r = n(p, x)$ and the assertion is proved.

By the above reason, for $\varepsilon > 0$ there is an integer $n(p, \varepsilon)$ such that $F_{f^n(p)f^{n+1}(p)}(\varepsilon) > 1-\lambda$ for all $n \geqq n(p, \varepsilon) = n_0$. If $f^{n_0}(p) = q$ we have that $F_{f^n(q)f^{n+1}(q)}(\varepsilon) > 1-\lambda$ for all integer $n \geqq n_0$. On the other hand for all $x > 0$

$$F_{f^n(q)f^{n+1}(q)}(x) \geqq F_{qf(q)}\left(\frac{x}{k^n}\right), \quad n = 0, 1, \dots$$

As in Theorem 2.2 we prove that the sequence $\{f^n(p)\}$ is fundamental in S. Let $f^n(p) \to p_0 \in S$ and $U_{f(p_0)}(\varepsilon', \lambda')$ be a neighborhood of $f(p_0)$. Since there is an integer $m \geqq 0$ such that $f^n(p) \in U_{p_0}(\varepsilon, \lambda) \cap U_{p_0}(\varepsilon', \lambda')$ for all $n \geqq m$ we have that

$$F_{f^{n+1}(p)f(p_0)}(\varepsilon') \geqq F_{f^n(p)p_0}\left(\frac{\varepsilon'}{k}\right) > 1-\lambda', \quad n \geqq m.$$

Therefore $f^n(p) \to f(p_0)$ and hence $f(p_0) = p_0$ and the existence of a fixed point of f is proved.

The uniqueness follows as in [12].

A similar result of EDELSTEIN [2] for nonarchimedean metric spaces is given by

Corollary 2. 2. *Let (S, d) be a complete ε-chainable metric space and $f\colon S \to S$ satisfy the condition that $d(p, q) < \varepsilon$ implies $d\big(f(p), f(q)\big) \leqq kd(p, q)$ for some $k, k \in (0, 1)$ and $p, q \in S$. Then f has a unique fixed point $p_0 \in S$ and $f^n(p) \to p_0$ for each $p \in S$.*

PROOF. Since for $F_{pq}(x) = H\big(x - d(p, q)\big)$, $p, q \in S$, $x \in R$ it follows that $(S, \mathscr{F}, \mathrm{Min})$ is a nonarchimedean (ε, λ)-chainable Menger space we observe that f is a (ε, λ)-local contraction for $\lambda > 1$ and the result follows.

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