On Cauchy's nucleus

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Let G and H be arbitrary groups (written additively) and $f: G \to H$ an arbitrary function. Let us denote

$$N_f = \{x \in G: \underset{y \in G}{\forall} (f(x+y) = f(x)+f(y))\}.$$

The set N_f will be called a *Cauchy nucleus*. This set is empty or it is a subgroup of G (cf. [1] Lemma 2 p. 313). [1])
The following problem has been put forward at the symposium on Functional Equations in Debrecen in 1973.
Does there exist for every groups G, H and every subgroup S of the group G a function $f: G \to H$ such that $N_f = S$. [2])

We are going to solve this problem in the present paper.

Theorem 1. *Let G be an arbitrary group and H an arbitrary group such that $2m \neq 0$ for some $m \in H$. Then for every subgroup S of the group G there exists a function $f: G \to H$ such that $N_f = S$.*

PROOF. Let us put

$$f(x) = \begin{cases} 0 & \text{for } x \in S, \\ m & \text{for } x \in G \setminus S. \end{cases}$$

We have

$$f(x+y) = 0 = f(x)+f(y) \quad \text{for} \quad x, y \in S$$

and

$$f(x+y) = m = O+m = f(x)+f(y) \quad \text{for} \quad x \in S, \quad y \in G \setminus S.$$

Thus $S \subset N_f$.
Let us consider an arbitrary $x \in G \setminus S$. Then $-x \in G \setminus S$ and consequently, from the definition of the function f, we obtain

$$0 = f(0) = f(x-x) \neq f(x)+f(-x) = m+m = 2m.$$

It means that $x \notin N_f$. Hence $S = N_f$, which completes the proof.

Theorem 2. *Let G be an arbitrary group and S its subgroup such that for every $x \in G \setminus S$ there exists $y \in G \setminus S$ such that $x+y \in G \setminus S$. Let H be an arbitrary non-*

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[1]) It has been proved in [1] p. 313 that the set $\{y \in G: \underset{x \in G}{\forall} (f(x+y) = f(x)+f(y))\}$ is empty or it is a subgroup of G. Proof for the set N_f can be realized in the similar way.

[2]) Genuinely, the problem has been formulated for $G = H$.

trivial group, i.e. such that card $H>1$. *Then there exists a function* $f: G\to H$ *such that* $N_f=S$.

PROOF. There exists an element $a\in H$ such that $a\neq 0$. We put

$$f(x)=\begin{cases}0 & \text{for } x\in S,\\ a & \text{for } x\in G\setminus S.\end{cases}$$

We have

$$f(x+y)=0=f(x)+f(y) \quad \text{for} \quad x\in S, \quad y\in S.$$

and

$$f(x+y)=a=0+a=f(x)+f(y) \quad \text{for} \quad x\in S, \quad y\in G\setminus S.$$

This proves that $S\subset N_f$. If $G=S$, then the proof is complete. Suppose that $S\neq G$. Let us consider an arbitrary element $x\in G\setminus S$. By the assumption there exists $y\in G\setminus S$ such that $x+y\in G\setminus S$. Consequently, we have

$$f(x+y)=a\neq a+a=f(x)+f(y).$$

Thus $x\notin N_f$. It proves that $N_f=S$, which completes the proof.

The assumption on the subgroup S in Theorem 2 can be replaced by a better known condition. We shall show it in the following

Theorem 3. *Let G be an arbitrary group and S its arbitrary subgroup. The following statements are equivalent:*
(A) *For every* $x\in G\setminus S$ *there exists* $y\in G\setminus S$ *such that* $x+y\in G\setminus S$,
(B) *The index of the subgroup S in the group G is different from two.*

PROOF. If index S equals two, then $x+y\in S$ for every $x\in G\setminus S$, $y\in G\setminus S$. It proves that condition (A) implies condition (B). Let us suppose for the converse implication that condition (A) does not hold. It means that there exists $x_0\in G\setminus S$ such that $x_0+y\in S$ for every $y\in G\setminus S$. But if $y\in G\setminus S$, then also $-y\in G\setminus S$. Consequently, we have for every $y\in G\setminus S$: $x_0-y\in S$, i.e. $y\in S+x_0$. It proves that $G\setminus S$ is a right coset of G with respect to subgroup S. Hence the index of S equals two, which completes the proof.

Theorems 1 and 2 do not solve the considered problem only in the case when index S equals two and $2x=0$ for every $x\in H$. In this case the answer to the set question may be negative.

In the next theorem we shall use the following result proved in [2] (Corollary 1).

Lemma 1. (cf. [2] Corollary 1). *Let G, H be arbitrary groups and let S be a subgroup of G. The general solution of the functional equation*[3])

(1) $$f(x+y)=f(x)+f(y) \quad \text{for} \quad x\in S, \quad y\in G,$$

where $f: G\to H$ *is the looked for function, has the following form:*

(2) $$f(x)=g(x-u)+h(u) \quad \text{for} \quad x\in S+U, \quad u\in U,$$

where

[3]) The following functional equation has been solved in note [2] $f(x\oplus y)=f(x)\oplus f(y)$ for $x\in G$, $y\in S$. Equation (1) can be reduced to this equation by putting $y\oplus x=x+y$.

$g\colon S\to H$ is an arbitrary homomorphism, $U\subset G$ is an arbitrary selector of the family $\{S+x\}_{x\in G}$ i.e. $U\cap\{S+x\}$ is a one-element set for every $x\in G$,
$h\colon U\to H$ is an arbitrary function satisfying condition

$$h(u_0)=g(u_0) \quad \text{for} \quad \{u_0\}=U\cap S. \tag{3}$$

Lemma 2. *Let H be a group. If $2x=0$ for every $x\in H$ then the group H is abelian.*
The proof is obvious.

Theorem 4. *Let G, H be groups and let $2m=0$ for every $m\in H$. Let S be a subgroup of G of the index two. Then the following statements are equivalent:*
(C) *There exists a function $f\colon G\to H$ such that $N_f=S$,*
(D) *There exists a normal subgroup T of the group S such that the group S/T is isomorphic to some subgroup H and for every $x\in G\setminus S$ there exists $y\in G\setminus S$ such that*

$$x+y-x-y\notin T, \tag{4}$$

or there exist $y\in G\setminus S$, $u\in G\setminus S$ such that

$$x-u+y-u-x\notin T. \tag{5}$$

Proof. Let us consider an arbitrary solution of the functional equation (1), i.e. a function $f\colon G\to H$ of the form (2), where U, u_0, u have the meanings as in Lemma 1. We have for $x\in S$:

$$f(x)=g(x-u_0)+h(u_0)=g(x)-g(u_0)+h(u_0)=g(x).$$

Since the index of S equals two, function f may be written as follows

$$f(x)=\begin{cases} g(x) & \text{for} \quad x\in S,\\ g(x-u)+h(u) & \text{for} \quad x\in G\setminus S, \quad \{u\}=(G\setminus S)\cap U. \end{cases} \tag{6}$$

Suppose that condition (C) holds. It means that there exists a solution f of equation (1) such that for every $x\in G\setminus S$ there exists $y\in G$ such that

$$f(x+y)\neq f(x)+f(y). \tag{7}$$

Two cases may occur: 1) $y\in S$, 2) $y\in G\setminus S$. In case 1)

$$x+y\in G\setminus S$$

and hence

$$f(x+y)=g(x+y-u)+h(u).$$

This equality implies that inequality (6) may be written in the following way:

$$g(x+y-u)+h(u)\neq g(x-u)+h(u)+g(y).$$

In virtue of Lemma 2 H is abelian. In consequence, the last inequality is equivalent to the following one

$$g(x+y-u)\neq g(y)+g(x-u)=g(y+x-u). \tag{8}$$

In case 1) $x+y\in S$ and in consequence inequality (7) receives the form

$$g(x+y)\neq g(x-u)+h(u)+g(y-u)+h(u).$$

But H is abelian and $2m=0$ for $m\in H$. Hence, the last inequality may be written in an iquivalent form as follows

(9) $$g(x+y) \neq g(x-u)+g(y-u) = g(x-u+y-u).$$

Let us put $T=\operatorname{Ker} g$. Obviously, S/T is isomorphic to the subgroup $g(S)$ of the group H. Inequality (8) is equivalent to the condition $x+y-u-(y+x-u)\notin T$ and hence it is equivalent to condition (4). Inequality (9) is equivalent to the condition (4). Inequality (9) is equivalent to the condition $x-u+y-u-(x+y)\notin T$ i.e. to condition (5). Thus, we have proved that if condition (C) is satisfied, so is condition (D).

Suppose now that condition (D) is fulfilled. Let t denote the isomorphism of the group S/T onto some subgroup of the group H and let $k\colon S\to S/T$ be the canonical mapping. We put $g=t\circ k$. Let U be any selector of the family $\{S, G\setminus S\}$ such that $\{u\}=(G\setminus S)\cap U$, where u is the element occuring in (5). Let $h\colon U\to H$ be an arbitrary function satisfying (3). Let us consider function f defined by (6). It is immediately seen that $T=\operatorname{Ker} g$. We have proved in the demonstration of the implication $(C)\Rightarrow(D)$ that the alternative of conditions (4) and (5), under respective notations, is equivalent to inequality (7). Hence, for every $x\in G\setminus S$ there exists $y\in G$ such that (7) holds. Because, in addition f satisfies (1) it proves that $N_f=S$, which completes the proof.

We obtain from Theorem 4 the following

Corollary. Let all assumptions of Theorem 4 be fulfilled and let additionally $2x=0$ for every $x\in G$. Then there does not exist a function $f\colon G\to H$ such that $N_f=S$.

PROOF. Let us observe that under our assumptions for every $x, y, u\in G$ and every subgroup T of the group S conditions (4) and (5) are not valid. This statement and Theorem 4 completes the proof.

We shall illustrate our considerations by the following

Example. Let G be the additive group of integers and $S\subset G$ a subgroup assembled from all even integers. As the group H we take the group $Z_2=\{0, 1\}$ with addition mod 2. Let $T\subset S$ be a subgroup of integers of the form $4n$ for n runs over the set of integers. It is obvious that the group S/T is isomorphic to the group H. Let us put $y=u=1$. Then $y, u\in G\setminus S$ and for every $x\in G$ $x-u+y-u-y-x=-2\notin T$. It means that condition (D) of Theorem 4 is satisfied. Hence, there exists a function $f\colon G\to H$ such that $N_f=S$. The construction of this function is given in the last part of the proof of Theorem 4 and in Lemma 1. In the considered case the function g occuring in this construction has the form

$$g(x) = \begin{cases} 0 & \text{for} \quad x = 4n, \quad n \in Z, \\ 1 & \text{for} \quad x = 4+2, \quad n \in Z. \end{cases}$$

In consequence, if we put $h(1)=0$ then we obtain (by this construction) the function f defined as follows

$$f(x) = \begin{cases} 0 & \text{for} \quad x = 4n, \quad n \in Z, \\ 1 & \text{for} \quad x = 4n+2, \quad n \in Z, \\ 0 & \text{for} \quad x = 4n+1, \quad n \in Z, \\ 1 & \text{for} \quad x = 4n+3, \quad n \in Z. \end{cases}$$

It can be also verified that for groups G, H, S there exist only two functions $f: G \to H$ such that $N_f = S$

References

[1] M. Kuczma, Cauchy's functional equation on a restricted domain, *Coll. Math.* **28** (1973), 313—315.
[2] J. Tabor, Solution of Cauchy's functional equation on a restricted domain, *Coll. Math. (to appear).*

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