The archimedean kernel of a lattice-ordered group

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0. Terminology and notation. Throughout this paper G will denote a lattice-ordered group (henceforth, l-group), written additively without regard to the commutativity of G. A subgroup H of G is called an *l-subgroup* if it is also a sublattice and is *convex* if $x\in H$ and $0<g\leqq x$ implies $g\in H$. If H is an l-subgroup of G, the *positive cone* of H, denoted H^+, is the set $\{h\in H|0\leqq h\}$. The *polar* of a subset $A\subseteq G$, denoted A', is the set $\{x\in G|\ |x|\wedge|a|=0$ for all $a\in A\}$ and we let $\{a\}'=a'$. A *value* for $0\neq x\in G$, denoted P_x, is a maximal element of the set of all convex l-subgroups of G which do not contain x. Let $\Gamma(G)$ be an index set for the set of all values of non-zero elements of G partially ordered by $\gamma\leqq\delta$ if $G_\gamma\subseteq G_\delta$. The intersection of all convex l-subgroups of G which properly contain a given value G_γ is the smallest convex l-subgroup properly containing G_γ and is called the *cover* of G_γ. The cover of G_γ is denoted G^γ. G is called *normal-valued* if each value is a normal subgroup of its cover. For $x\in G$, $G(x)$ will denote the convex l-subgroup generated by x. For other terminology and notation, the reader is referred to [7] or [9].

1. The existence of the archimedean kernel. Our original proof of the existence of the archimedean kernel required that G be representable ([10]). If $G(g)$ is archimedean, $G(g)$ is contained in the largest abelian convex l-subgroup of G, so the existence of the archimedean kernel for representable l-groups implies the existence in general. In [13], REDFIELD proved the existence of the archimedean kernel for an arbitrary l-group. Since Redfield's proof is easier, we will use it here.

An element $a\in G^+$ is called *archimedean* if for each $0<g\leqq a$, there exists an integer $n>0$ such that $ng\nleqq a$. Let $P(G)$ be the set of all archimedean elements of G^+ and let $A(G)$ be the convex l-subgroup generated by $P(G)$.

Theorem 1.1. (REDFIELD, [13]). $A(G)^+=P(G)$.

PROOF. Since $P(G)$ is a convex normal subset of G which contains 0, it suffices to show $P(G)$ is a subsemigroup of G^+. Suppose (by way of contradiction) that there exists $a, b\in P(G)$ such that $a+b\notin P(G)$. Then there is a $0<t\leqq a+b$ such that $nt\leqq a+b$ for all integers $n>0$. Since a is archimedean, there exists an integer $m>0$ such that $mt\nleqq a$ ($m=1$ is possible). Then

$$s=(-a+mt)\vee 0>0.$$

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Since $nt \leqq a+b$ for all integers $n>0$,

$$(1) \qquad -a+nt \leqq b \quad \text{for all} \quad n>0,$$

so

$$(2) \qquad (-a+nt)\vee 0 \leqq b \quad \text{for all} \quad n>0.$$

We will show by induction on k that

$$ks \leqq (-a+kmt)\vee 0 \quad \text{for all} \quad k>0.$$

The case when $k=1$ is valid by the definition of s. Suppose that the inequality is valid for $k=h$, i.e.,

$$(3) \qquad hs \leqq (-a+hmt)\vee 0.$$

Then,

$$(h+1)s = hs+s \leqq (-a+hmt)\vee 0+(-a+mt)\vee 0$$

by (3). Since $-a+hmt \geqq -a+mt$ and $-a+hmt+mt \geqq -a+hmt-a+mt$,

$$(h+1)s \leqq (-a+hmt-a+mt)\vee(-a+hmt)\vee(-a+mt)\vee 0 \leqq$$

$$\leqq (-a+hmt+mt)\vee(-a+hmt)\vee 0.$$

Therefore,

$$(h+1)s \leqq \big(-a+(h+1)mt\big)\vee(-a+mt)\vee 0 \leqq \big(-a+(h+1)mt\big)\vee 0,$$

which completes the induction. Now, by (2),

$$ks \leqq (-a+kmt)\vee 0 \leqq b$$

for all $k>0$ which is impossible since b is archimedean. Therefore, $a+b$ is archimedean and so $S(G)=A(G)^+$.

Corollary 1.2. (Redfield, [13]) $A(G)$ is the (unique) largest convex archimedean l-subgroup of G.

An element $0 \neq s \in G^+$ is called *basic* if the set $\{x \in G^+ \mid x \leqq s\}$ is totally ordered. G has a *basis* if each positive element exceeds a basic element. If $s \in G$ is basic, then s' is a prime convex l-subgroup of G ([17]) so s has a unique value. A convex l-subgroup K of G is said to be *closed* if for each subset $\{h_\lambda \mid \lambda \in B\}$ of K such that $h=\vee h_\lambda$ exists in G, then $h \in K$.

Theorem 1.3. *Let G be normal-valued and let $0 \neq x \in G$. The following are equivalent:*

(a) $x \in A(G)$.

(b) $|x|$ *is archimedean.*

(c) $G(|x|)$ *is an archimedean l-group.*

(d) $x' = \bigcap \{P_x \mid P_x$ *is a value for* $x\}$.

If, in addition, G has radical zero (*see* [7]) *and Δ is the minimal plenary subset of $\Gamma(G)$, each of the above is equivalent to*

(e) *If $\delta \in \Delta$ is a value for x, then δ is a minimal element of $\Gamma(G)$.*

PROOF. The equivalence of (a), (b), and (c) can be found in [13].

(c)→(d) Clearly $x' \subseteqq \cap P_x$. Let $y \in \cap P_x$. Then

$$|x| \wedge |y| \in G(|x|) \cap (\cap P_x) = \cap (P_x \cap G(|x|)) = \{\text{maximal } l\text{-ideals of } G(|x|)\} = 0$$

since $G(|x|)$ is an archimedean l-group with a strong order unit. Thus $y \in x'$ and $x' = \cap P_x$.

(d)→(b) Suppose (by way of contradiction) there exists $0 < t \leqq x$ such that $nt \leqq x$ for all integers $n > 0$. Since $0 \neq t = t \wedge |x|$, by (d), there is a value P_x of x such that $t \notin P_x$. Since $|x| \geqq t > 0$, P_x is also a value for t. Now

$$|x| + P_x \geqq nt + P > P_x \quad \text{for all integers} \quad n > 0$$

which is impossible since G is normal-valued. Thus $|x|$ is archimedean.

Now suppose G has radical zero and Δ is the minimal plenary subset of $\Gamma(G)$.

(a)→(e) Since G has radical zero, so does $A(G)$ and so $A(G)$ has a basis, (see [7]), say $S = \{s_\lambda | \lambda \in \Lambda\}$. Then an isomorphic copy of $A(G)$ lies between $\Sigma_\Lambda G(s_\lambda)$ and $\Pi_\Lambda G(s_\lambda)$. For each $\lambda \in \Lambda$, let $G_\lambda = s'_\lambda$ be the unique value of s_λ, (since $s_\lambda \in A(G)$, $s'_\lambda = \cap \{P | P \text{ is a value of } s_\lambda\} = G_\lambda$). Since G_λ is a polar, it is a minimal prime. Let $0 \neq x \in A(G)$ and let $\delta \in \Delta$ be a value for x. Since G_δ is essential (see [7]), it is closed, so there exists a $\lambda \in \Lambda$ such that $s_\lambda \notin G_\delta$. Therefore, $G_\delta = G_\lambda$ and δ is a minimal element of $\Gamma(G)$.

(e)→(b) Let $x \in G$ be such that each value for x in Δ is a minimal element of $\Gamma(G)$. Suppose (by way of contradiction) there exists $0 < t \in G$ such that $nt \leqq |x|$ for all integers $n > 0$. Let $\delta \in \Delta$ be a value for t. Since $x \notin G$ and each value for x in Δ is minimal, δ is a value for x. But then, $|x| + G_\delta \geqq nt + G_\delta > G_\delta$ which is impossible since G is normal-valued. Thus $|x|$ is archimedean.

Remark. The proof of theorem 1.1 shows that (e)→(a)↔(b)↔(c)→(d) is valid for arbitrary l-groups.

2. Properties of $A(G)$. If G is an l-subgroup of H, then H is called an a^*-extension of G if $K \to K \cap G$ is a one-to-one map of the closed convex l-subgroups of H onto those of G.

G is said to be an

L-group if $\vee s_\lambda$ exists for any disjoint subset $\{s_\lambda\}$ of G.

P-group if $G = g' \oplus g''$ for all $g \in G$.

SP-group if $G = A' \oplus A''$ for all $A \subseteqq G$.

O-group if G is an L-group and a P-group.

Let G be an l-subgroup of H. H is called an *X-hull* of G, for $X = P$, SP, L or O, if G is large in H, H is an X-group and no proper l-subgroup of H which contains G is an X-group. The X-hull of G will be denoted by G^X. Notice that if $X = P$, SP or O, then G^X, and hence G, is representable. For discussion of X-hulls, see [8], and for the existence of L-hulls, see [1].

Proposition 2.1.

(a) $A(G)$ is a closed l-characteristic l-subgroup of G.

(b) Let K be an l-subgroup of G. If K is either large or convex in G, then $A(K) = K \cap A(G)$.

(c) If G is an X-group, so is $A(G)$ for $X = P$, SP, L or O.

(d) $A(G)^X \subseteqq A(G^X)$ but equality need not hold for $X=P$ or SP even if G is abelian with a basis.

(e) $A(\Pi G_\lambda)=\Pi A(G_\lambda)$.

PROOF. (a) Clearly $A(G)$ is l-characteristic. BLEIER and CONRAD ([2] and [3]) have shown that the closure of a convex l-subgroup is an a^*-extension and that an a^*-extension of an archimedean l-group is archimedean. Thus, maximality of $A(G)$ implies that it is closed.

(b) If K is convex in G, the result is clear. $A(K)\supseteqq A(G)\cap K$ is true for any l-subgroup K of G. Suppose K is large in G and let $0<x\in A(K)$. Let $g\in G$ be such that $0<g<x$. Since K is large in G, there is an integer $n>0$ and a $k\in K$ such that $0<k<ng$. Since $x\in A(K)$, there is an integer $m>0$ such that $mk \not\leqq x$ ($m=1$ is possible). Then $mng \not\leqq x$ so $x\in A(G)$.

(c) By [8], a closed convex l-subgroup of an X-group is an X-group.

(d) Since G is large in G^X, $A(G)$ is large in $A(G^X)$, and by (c), $A(G^X)$ is an X-group. The intersection of all l-subgroups of $A(G^X)$ which contain $A(G)$ and are X-groups is the X-hull of $A(G)$. Thus $A(G)^X \subseteqq A(G^X)$. For the last statement, see example 1 in section 4.

(e) Clear.

Let K be an l-subgroup of G. K is said to be an *$\mathcal{L}$-subgroup* of G if, for each set $\{u_\beta|\beta\in B\}$ of disjoint elements of K such that $\bigvee_K u_\beta$ exists, it follows that $\bigvee_G u_\beta$ exists and equals $\bigvee_K u_\beta$. An *H-representation* of G is a pair $(\sigma, \Pi H_\lambda)$ where σ is an l-isomorphism of G onto a subdirect sum of ΠH_λ and each H_λ is a transitive l-subgroup of the l-group of all permutations of a totally ordered set T_λ. An H-representation is called *complete* if σ preserves all joins and intersections existing in G. BYRD and LLOYD, [4], have shown that G is completely distributive if and only if G has a complete H-representation $(\sigma, \Pi H_\lambda)$ and, [5], the intersection of all laterally complete $\mathcal{L}$-subgroups of ΠH_λ which contain $G\sigma$ is the L-hull of G.

Theorem 2.2. *If G is completely distributive, then* $A(G)^L=A(G^L)$.

PROOF. Let $(\sigma, \Pi H_\lambda)$ be a complete H-representation of G. By abuse of notation, we will suppress σ and view G as an l-subgroup of ΠH_λ. Now, $A(G)$ is completely distributive and so has a basis, say $S=\{s_\beta|\beta\in B\}$. S is also a basis for $A(G^L)$. Let $0<x\in A(G^L)$. For each $\beta\in B$, there is an integer $n_\beta>0$ such that $n_\beta s_\beta \not\leqq x$. Let $x_\beta=n_\beta s_\beta\wedge x$. Then $x_\beta\wedge x_\gamma=0$ for $\beta\neq\gamma$ and $x=\bigvee x_\beta$. Thus, in order to show $A(G)^L=A(G^L)$, we need only show $x_\beta\in G$. If $x_\beta=0$, we are done. Suppose $x_\beta>0$. Then there is a λ such that $(x_\beta)_\lambda>0$. Since G is a subdirect sum of ΠH_λ, there is a $0<g\in G$ such that $g_\lambda=(x_\beta)_\lambda$. By replacing g by $n_\beta s_\beta\wedge g$, we may assume $g\in G(s_\beta)$. Since s_β is basic, $G^L(s_\beta)$ is totally ordered and so a subgroup of the real numbers, since $s_\beta\in A(G^L)$. Since $g_\lambda=(x_\beta)_\lambda$, we have $ng>mx_\beta>0$ if $n>m>0$, and $mx_\beta>ng>0$ if $m>n>0$. Thus $g=x_\beta$ and $A(G^L)=A(G)^L$.

Since the class of archimedean l-groups is not closed under l-homomorphic images, $A(G)\beta$ need not be contained in $A(G\beta)$ for an arbitrary l-homomorphism β. It is natural to ask when $A(G)\beta$ is contained in $A(G\beta)$ for all l-homomorphisms β. If $A(G)$ is hyperarchimedean, $A(G)\beta\subseteqq A(G\beta)$ is always valid. Conversely, if G is representable and $A(G)$ is not hyperarchimedean, let P be a minimal but not maximal prime l-ideal of $A(G)$. Then $A(G)/P$ is not archimedean. Now, let Q be the minimal prime l-subgroup of G such that $Q\cap A(G)=P$. Since G is representable, Q is normal in G and $(A(G)+Q)/Q \simeq A(G)/P$ is not archimedean. Thus, if G is representable, $A(G)\beta \subseteq A(G\beta)$ for all l-homomorphisms β if and only if $A(G)$ is hyperarchimedean. The following theorem gives an indication of what may happen if G is not representable.

Theorem 2.3. (1) *Let G be an l-group such that for every l-homomorphism β, $A(G)\beta \subseteq A(G\beta)$. Then for each characteristic convex l-subgroup K of $A(G)$, $A(G)/K$ is archimedean.*

(2) *Conversely, if A is an archimedean l-group such that for each characteristic convex l-subgroup K of A, A/K is archimedean, then there is an l-group G such that $A(G) \simeq A$ for every l-homomorphism β, $A(G)\beta \subseteq A(G\beta)$.*

PROOF. (1) Since $A(G)$ is characteristic in G, and K is characteristic in $A(G)$, K is normal (in fact, characteristic) in G. Thus $A(G)/K \subseteq A(G/K)$ so $A(G)/K$ is archimedean.

(2) Conversely, suppose A is an archimedean l-group such that A/K is archimedean for all characteristic convex l-subgroups K of A. Let $\mathfrak{A}$ be the l-automorphism group of A and let F be a free group with epimorphism $\mu\colon F \to \mathfrak{A}$. Since F is free, it is orderable (see [9], page 49) so we will view F as an ordered group. Let G be the splitting extension of A by F via μ, lexicographically ordered. I.e., the underlying set is $F \times A$ with group operation $(f, a)+(g, b)=(f+g, a(g\mu)+b)$ ordered by $(f, a) \geqq 0$ if $f>0$ or $f=0$ and $a \geqq 0$. It is easy to show that G is an l-group. $A(G) \simeq A$, and if P is an l-ideal of G, $P \cap A(G)$ is a characteristic convex l-subgroup of $A(G)$ so $A(G)\beta \subseteq A(G\beta)$ for all l-homomorphisms β.

Since the archimedean kernel does not have many of the nice properties of the hyperarchimedean kernel, (see [12])], we also considered the possibility of a maximal subdirect product of subgroups of the totally ordered group of real numbers. Unfortunately, as example 2 in section 4 shows, this does not, in general exist.

3. The Archimedean kernel sequence. Following the construction of the ascending central series for groups, we have

$$A(G) = A^1(G) \subseteqq A^2(G) \subseteqq A^3(G) \subseteqq \dots$$

where, for an ordinal β,

$$A^{\beta+1}(G)/A^\beta(G) = A(G/A^\beta(G)),$$

and if β is a limit ordinal,

$$A^\beta(G) = \bigcup_{\mu<\beta} A^\mu(G).$$

Let $a, b \in G^+$. We say a is *archimedean less than* b, denoted $a \ll b$, if $na \leqq b$ for all integers $n>0$. Thus, $0<a \in A(G)$ if and only if $a \gg b$ implies $b=0$. It seems reasonable to suppose $0<a\in A^2(G)$ if and only if $a \gg b \gg c$ implies $c=0$. In fact, it is easy to show that if $0<a\in A^2(G)$ then $a\gg b\gg c$ does imply $c=0$, but the converse is false, as example 3 shows. We do not know if there is an elementwise characterization of $A^2(G)$.

By a simple cardinality argument, there is an ordinal β (depending on G) so that $A^\beta(G)=A^{\beta+1}(G)$. Let $A^*(G)=A^\beta(G)$. We now investigate conditions for $A^*(G)=G$.

Theorem 3.1. $A^*(G)\subseteqq[A(G)]''$. Thus if $A^*(G)=G$, then $A(G)$ is dense in G.

PROOF. It suffices to show $A^\gamma(G)\cap[A(G)]'=0$ for each ordinal γ. We will induct on γ. The case when $\gamma=1$ or γ a limit ordinal is clear. Suppose $A^{\gamma-1}(G)\cap\cap[A(G)]'=0$ and let $0\leqq x\in A^\gamma(G)\cap[A(G)]'$. Suppose (by way of contradiction) $x\neq 0$. Then $x\notin A^{\gamma-1}(G)\supseteqq A(G)$, so there is a $0<y\ll x$. Thus $x+A^{\gamma-1}(G)\gg\gg y+A^{\gamma-1}(G)$ so $y\in A^{\gamma-1}(G)$. Since $0<y<x$, $y\in[A(G)]'$ so $y\in[A(G)]'\cap A^{\gamma-1}(G)=0$, a contradiction. Thus $A^\gamma(G)\cap[A(G)]'=0$.

This condition is not sufficient since if $H=V(\Delta, R)$ (see [7]) where Δ is totally ordered and R is the totally ordered group of real numbers, than $A(H)=0$ if and only if Δ has no minimal element. Let B be an archimedean l-group. The lex-extension G of B by H is an l-group with $A(G)\simeq B$ dense in G but if Δ has no minimal element, $A^2(G)=A(G)\neq G$.

Lemma 3.2. *If K is a convex l-subgroup of G, then $K\cap A^*(G)=A^*(K)$.*

The proof of this is similar to the proof of Proposition 1.4 of [10] and is omitted.

Theorem 3.3. *Let G be a normal-valued l-group and suppose $\Gamma(G)$ satisfies the descending chain condition. Then $A^*(G)=G$.*

PROOF. We will show that the descending chain condition on $\Gamma(G)$ forces each sequence of the form

$$x_1\gg x_2\gg\dots,\ x_i>0$$

to be finite. This will guarantee that $A(G)\neq 0$ and a simple cardinality argument will show $A^\beta(G)=G$. Suppose (by way of contradiction) that there is an infinite sequence

$$x_1\gg x_2\gg x_3\gg\dots,\ x_i>0. \quad (\omega)$$

Since G is normal-valued, each value of x_{i+1} is properly contained in a value of x_i. The set $\{x_i|1\leqq i\}$ is a filter in G^+ and hence is contained in an ultrafilter $\mathscr{J}$. $G^+\setminus\mathscr{J}$ is the positive cone of a prime l-subgroup P. Since the collection of all convex l-subgroups which contain P is totally ordered, and $x_i\notin P$ for all i, there is a chain

$$G_1\supset G_2\supset G_3\supset\dots,$$

where G_i is a value of x_i and each containment is proper. This contradicts the fact that $\Gamma(G)$ satisfies the descending chain condition. Therefore any sequence of the form (ω) is finite.

This condition is not necessary since an infinite cardinal product of the l-group of real numbers is an archimedean l-group and has minimal primes which are not values and therefore does not satisfy the descending chain condition, so $A(G)=G$ but $\Gamma(G)$ does not satisfy the descending chain condition.

4. Examples. 1. An example of an abelian l-group G with a basis such that $A(G)^X\neq A(G^X)$ for $X=P$ or SP.

Let G be the set of all real-valued sequences with domain the non-negative integers such that $f\in G$ implies there exists an n so that if $m\geqq n$, then $f(0)=f(m)$. When G has pointwise addition and partial order $f\geqq g$ if $f(0)>g(0)$ and $f(n)\geqq g(n)$ for $n>1$ or $f(0)=g(0)$ and $f(n)\geqq g(n)$ for $n\geqq 1$, then G is an l-group. Then

$A(G)=\{f\in G\,|\,f(0)=0\}$ which is an SP-group. G^{SP} is the l-group of bounded functions and G^{P} is the l-group of eventually constant sequences. $A(G^{SP})=\{f\in G^{SP}|\, f(0)=0\}$ and $A(G^{P})=\{f\in G^{P}\,|\,f(0)=0\}$.

2. An example of an archimedean l-group G which has no convex l-gubgroup which is maximal with respect to being a subdirect product of subgroups of the l-group of real numbers. This example is due to SHELDON [14].

Let F be the l-subgroup of $\prod_{[0,1]} Z$ generated by the characteristic functions of closed intervals, where Z is the integers. For each $b\in[0, 1]$, let

$$h_b(x)=\begin{cases}[1/(x-b)^2] & x\neq b\\ 0 & x=b\end{cases}$$

where [...] is the greatest integer function. Let K be the subgroup of ΠZ generated by the h_b, $b\in[0, 1]$ and F. Then any $f\in K$ can be written in the form

$$f=k+n_1h_{b_1}+n_2h_{b_2}+\ldots+n_mh_{b_m}\quad (\zeta)$$

where $k\in F$ and each $n_j\neq 0$. Let $G=K/\Sigma Z$. Then
1. G is an l-subgroup of $\Pi Z/\Sigma Z$.
2. G is archimedean.
3. The prime l-ideals of G are

$$P_b=\{f+\Sigma Z\in G\,|\,h_b \text{ does not occur in the expansion } (\zeta) \text{ for } f.\}$$

$$P_b^l=\{f+\Sigma Z\in G\,|\,f(x)=0 \text{ for almost all } x\in(y, b) \text{ some } y<b\},\ (b\neq 0)$$

$$P_b^r=\{f+\Sigma Z\in G\,|\,f(x)=0 \text{ for almost all } x\in(b, z) \text{ some } z>b\},\ (b\neq 1)$$

4. $P_b\supseteqq P_b^l\cup P_b^r$.
5. $F/\Sigma Z\subseteqq \cap P_b$, so G is not a subdirect sum of subgroups of the real numbers.
6. If H is a convex l-subgroup of G, then H is a subdirect sum of subgroups of the reals if and only if
$S=\{b\in[0, 1]\,|\,P_b\cap H\subset H$ has empty interior$\}$.

1—5 are in [14].

Proof of 6. Since H is convex, $\Gamma(H)$ is precisely $\{K\in\Gamma(G)\,|\,K\cap H\subset H\}$.

($\rightarrow$) Suppose S has non-empty interior, say $(s, t)\subseteqq S$. Let $s<m<n<t$. Since $[m, n]\subseteqq(s, t)\subseteqq S$, we can find a $g\in H$ so that $g(x)\geqq 1$ for all $x\in[m, n]$, and $g(x)=0$ if $x\notin(s, t)$. Now this g is in every maximal prime of H so H is not a subdirect sum of real numbers.

($\leftarrow$) Suppose (by way of contradiction) S has empty interior. If $\Sigma Z<\Sigma Z+g\in H$, then there is an interval $(s, t)\subset[0, 1]$ such that $g(x)\geqq 1$ for almost all $x\in(s, t)$. Since S has empty interior, there is a $b\in(s, t)$ such that $b\notin S$, so $P_b\cap H=H$. Since $g(x)\geqq 1$ for almost all $x\in(s, b)$, $g+\Sigma Z\notin P_b^R$ which is a maximal prime of H. Thus H is a subdirect product of subgroups of R.

Since [0, 1] has no subsets maximal with respect to having empty interior, G has no maximal subdirect product of subgroups of R.

3. An example of an l-group such that for all $a\in G$, $a\gg b\gg c$ implies $c=0$, but $A^2(G)\neq G$.

Let G be the set of all real sequences $(a_0, a_1, a_2, \ldots)$ such that $a_n=na_0+a_1$ for all but finitely many $n \geqq 2$ with pointwise addition and ordered by $(a_0, a_1, a_2, \ldots) \geqq 0$ if $a_n \geqq 0$ for $n > 2$ *and* $a_0 > 0$, or $a_0 = 0$ and $a_1 > 0$, or $a_0 = a_1 = 0$ and $a_2 \geqq 0$. It is easy to see that $A(G)$ is the set of all sequences with $a_0 = a_1 = 0$ and $G/A(G) = \overrightarrow{R \oplus R}$.

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