On birecurrent Kähler manifold

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In this paper we have studied the properties of Bochner curvature tensor in a Kähler manifold by making use of birecurrent condition.

1. *Introduction.*

Agreement 1.1. In what follows the equations will hold for arbitrary C^∞ vector fields $X, Y, Z, T, W, \dots,$ etc. whenever they occur.

We consider a $2n$-dimensional real manifold M_{2n} of differentiability class C^{r+1}. Let F be the vector valued linear function defined in M_{2n}, such that

$$\overline{X}+X=0, \tag{1.1}$$

for arbitrary vector field X in M_{2n}, where

$$\overline{X} \overset{\text{def}}{=} F(X). \tag{1.2}$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an almost complex manifold.

If the Hermite metric g:

$$g(\overline{X}, \overline{Y}) = g(X, Y), \tag{1.3}$$

be defined in almost complex manifold, then M_{2n} is called an almost Hermite manifold.

Let us put

$$'F(X, Y) = g(\overline{X}, Y). \tag{1.4}$$

Then from (1.1), (1.2), (1.3), and (1.4), we have

$$'F(\overline{X}, \overline{Y}) = {}'F(X, Y), \tag{1.5a}$$

$$'F(X, Y)+{}'F(Y, X) = 0. \tag{1.5b}$$

Suppose D is a Riemannian connexion satisfying:

$$D_X Y - D_Y X = [X, Y], \tag{1.6a}$$

$$(D_X g)(Y, Z) = g(D_X Y, Z)+g(Y, D_X Z). \tag{1.6b}$$

If in addition to (1.2), (1.3) and (1.6) the condition

$$(D_X F)(Y) = 0 \tag{1.7}$$

is satisfied, then M_{2n} is called Kähler manifold.

Let K be the curvature tensor of M_{2n} given by

(1.8) $$K(X, Y, Z) \overset{\text{def}}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z.$$

Let Ric be the Ricci tensor of M_{2n} given by

(1.9) $$\mathrm{Ric}(Y, Z) \overset{\text{def}}{=} (C_1' K)(Y, Z),$$

and

(1.10) $$\mathrm{Ric}(Y, Z) \overset{\text{def}}{=} \mathrm{Ric}(Z, Y).$$

Let us put

(1.11) $$\mathrm{Ric}(Y, Z) \overset{\text{def}}{=} g\big(r(Y), Z\big)$$

and

(1.12) $$R \overset{\text{def}}{=} (C_1' r),$$

where R is the scalar curvature.

We know that a manifold M_{2n} is called birecurrent manifold (Takeno, 1971) if

(1.13) $$(\nabla\nabla K)(Z, T, W, X, Y) = a(X, Y) K(Z, T, W),$$

where $a(X, Y)$ is a non-vanishing C^∞ function.

The manifold M_{2n} is called Ricci birecurrent if

(1.14)a $$(\nabla\nabla \,\mathrm{Ric})(Z, T, X, Y) = a(X, Y)\,\mathrm{Ric}(Z, T),$$

which implies

(1.14)b $$(\nabla\nabla r)(Z, X, Y) = a(X, Y) r(Z),$$

and

(1.14)c $$(\nabla\nabla R)(X, Y) = a(X, Y) R,$$

where

(1.14)d $$(D_X K)(Z, T, W) = (\nabla K)(Z, T, W, X).$$

The holomorphic sectional curvature K of M_{2n} with regard to X is given by (Mishra, [1])

(1.15) $$Kg(X, X) g(X, X) + K(X, \bar{X}, X, \bar{X}) = 0.$$

The projective curvature tensor W^*, the conformal curvature tensor V, the conharmonic curvature tensor L, the concircular curvature tensor C, H-projective curvature tensor P, T-concircular curvature tensor T^* are given by

(1.16) $$W^*(Z, T, W) = K(Z, T, W) - \frac{1}{(2n-1)}[\mathrm{Ric}(T, W) Z - \mathrm{Ric}(Z, W) T],$$

(1.17) $$V(Z, T, W) =$$

$$= K(Z, T, W) - \frac{1}{(2n-1)}[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T + r(Z)g(T, W) - r(T)g(Z, W)] +$$

$$+ R/2(2n-1)(n-1)[g(T, W)Z - g(Z, W)T],$$

(1.18) $$L(Z, T, W) =$$

$$= K(Z, T, W) - \frac{1}{2(n-1)}[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T + r(Z)g(T, W) - r(T)g(Z, W)],$$

(1.19) $$C(Z, T, W) = K(Z, T, W) - \frac{R}{2n(2n-1)}[g(T, W)Z - g(Z, W)T],$$

(1.20) $$P(Z, T, W) = K(Z, T, W) -$$

$$- \frac{1}{2(n+1)}[\mathrm{Ric}(T, W)Z - \mathrm{Ric}(Z, W)T + \mathrm{Ric}(\overline{T}, W)\overline{Z} + \mathrm{Ric}(Z, \overline{W})\overline{T} + 2\,\mathrm{Ric}(Z, \overline{T})\overline{W}],$$

(1.21) $$T^*(Z, T, W) =$$

$$= K(Z, T, W) + \frac{R}{4n(n+1)}[g(Z, W)T - g(T, W)Z + g(\overline{Z}, W)\overline{T} - g(\overline{T}, W)\overline{Z} + g(\overline{Z}, T)\overline{W}],$$

respectively.

Definition 1.1. Let Q be a vector valued trilinear function, by any one of the curvature tensor W^*, V, L, C, P, T^*. Then the Kähler manifold M_{2n} is said to be Q-birecurrent (RATHORE and MISHRA, 1973) if

(1.22) $$(\nabla\nabla Q)(Z, T, W, X, Y) = a(X, Y)Q(Z, T, W).$$

2. *The Bochner Curvature tensor.*

The Bochner curvature tensor B in Kähler manifold M_{2n} is given by

(2.1)

$$B(Z, T, W) = K(Z, T, W) + \frac{1}{2(n+2)}[\mathrm{Ric}(Z, W) - \mathrm{Ric}(T, W)Z + g(Z, W)r(T) -$$

$$- g(T, W)r(Z) + \mathrm{Ric}(\overline{Z}, W)\overline{T} - \mathrm{Ric}(\overline{T}, W)\overline{Z} + g(\overline{Z}, W)r(\overline{T}) - g(\overline{T}, W)r(\overline{Z}) +$$

$$+ 2\,\mathrm{Ric}(\overline{Z}, T)\overline{W} + 2g(\overline{Z}, T)r(\overline{W})] -$$

$$- \frac{R}{4(n+1)(n+2)}[g(Z, W)T - g(T, W)Z + g(\overline{Z}, W)\overline{T} - g(\overline{T}, W)\overline{Z} + 2g(\overline{Z}, T)\overline{W}].$$

Let us put

(2.2) $$'B(Z, T, W, U) \stackrel{\mathrm{def}}{=} g(B(Z, T, W), U).$$

Then we have

(2.3)

$$'B(Z, T, W, U) = 'K(Z, T, W, U) + \frac{1}{2(n+2)}[g(T, U)\,\mathrm{Ric}\,(Z, W) - g(Z, U)\,\mathrm{Ric}\,(T, W) +$$
$$+ g(Z, W)\,\mathrm{Ric}\,(T, U) - g(T, W)\,\mathrm{Ric}\,(Z, U) + \mathrm{Ric}\,(\overline{Z}, W)g(\overline{T}, U) - \mathrm{Ric}\,(\overline{T}, W)g(\overline{Z}, U) +$$
$$+ g(\overline{Z}, W)\,\mathrm{Ric}\,(\overline{T}, U) - g(\overline{T}, W)\,\mathrm{Ric}\,(\overline{Z}, U) +$$
$$+ 2\,\mathrm{Ric}\,(\overline{Z}, T)g(\overline{W}, U) + 2g(\overline{Z}, T)\,\mathrm{Ric}\,(\overline{W}, U)] -$$
$$- \frac{R}{4(n+1)(n+2)}[g(Z, W)g(T, U) - g(T, W)g(Z, U) + g(\overline{Z}, W)g(\overline{T}, U) -$$
$$- g(\overline{T}, W)g(\overline{Z}, U) + 2g(\overline{Z}, T)g(\overline{W}, U)].$$

Definition 2.1. The Kähler manifold M_{2n} will be called a Bochner birecurrent manifold if

$$(\nabla\nabla B)(Z, T, W, X, Y) + a(X, Y)B(Z, T, W). \tag{2.4}$$

Now we have following theorems:

Theorem 2.1. *If a Kähler manifold is a Bochner birecurrent manifold and a Ricci birecurrent manifold for the same recurrence parameter, it is also a birecurrent manifold.*

PROOF. From (2.1), we have

(2.5)

$$(\nabla\nabla B)(Z, T, W, X, Y) = (\nabla\nabla K)(Z, T, W, X, Y) + \frac{1}{2(n+2)}[(\nabla\nabla\,\mathrm{Ric})(Z, W, X, Y)T -$$
$$-(\nabla\nabla\,\mathrm{Ric})(T, W, X, Y)Z + (\nabla\nabla r)(T, X, Y)g(Z, W) - (\nabla\nabla r)(Z, X, Y)g(T, W) +$$
$$+ (\nabla\nabla\,\mathrm{Ric}\,(\overline{Z}, W, X, Y)\,\overline{T} - (\nabla\nabla\,\mathrm{Ric})(\overline{T}, W, X, Y)\overline{Z} + (\nabla\nabla r)(\overline{T}, X, Y)g(\overline{Z}, W) -$$
$$- (\nabla\nabla r)(\overline{Z}, X, Y)g(\overline{T}, W) + 2(\mathrm{Ric})(\overline{Z}, T, X, Y)\overline{W} + 2(\nabla\nabla r)(\overline{W}, X, Y)g(\overline{Z}, T)] -$$
$$- \frac{(\nabla\nabla R)(X, Y)}{4(n+1)(n+2)}[g(Z, W)T - g(T, W)Z + g(\overline{Z}, W)\,\overline{T} - g(\overline{T}, W)\overline{Z} + 2g(\overline{Z}, T)\overline{W}].$$

Let the Kähler manifold M_{2n} be Ricci birecurrent and Bochner birecurrent manifold. Then by using (1.14)a, (1.14)b, (1.14)c and (2.4) in (2.5) we get

$$(\nabla\nabla K)(Z, T, W, X, Y) = \tag{2.6}$$

$$= a(X, Y)\big[B(Z, T, W) - \frac{1}{2(n+2)}\{\mathrm{Ric}\,(Z, W)T - \mathrm{Ric}\,(T, W)Z + r(T)g(Z, W) -$$
$$- r(Z)g(T, W) + \mathrm{Ric}\,(\overline{Z}, W)\overline{T} - \mathrm{Ric}\,(\overline{T}, W)\overline{Z} + r(\overline{T})g(\overline{Z}, W) - r(\overline{Z})g(\overline{T}, W) +$$
$$+ 2\,\mathrm{Ric}\,(\overline{Z}, T)\overline{W} + 2r(\overline{W})g(\overline{Z}, T)\} -$$
$$- \frac{R}{4(n+1)(n+2)}\{g(Z, W)T - g(T, W)Z + g(\overline{Z}, W)\overline{T} - g(\overline{T}, W)\,\overline{Z} + 2g(\overline{Z}, T)\overline{W}\}\big].$$

Using equation (2.1) in the above equation, we have

$$(2.7)\qquad (\nabla\nabla K)(Z,T,W,X,Y)=a(X,Y)K(Z,T,W),$$

that is, Kähler manifold M_{2n} is a birecurrent manifold, which proves the statement.

Similarly, we can prove that if a Kähler manifold is birecurrent and either Bochner birecurrent or Ricci birecurrent manifold for the same recurrence parameter, i.e. also either Ricci birecurrent or Bochner birecurrent manifold respectively.

Theorem 2.2. *We have*

$$\begin{aligned}B(Z,T,W)&=\dot{W}^*(Z,T,W)+\frac{5}{2(2n-1)(n+2)}[\mathrm{Ric}(T,W)Z-\mathrm{Ric}(Z,W)T]+\\&+\frac{1}{2(n+2)}[g(Z,W)r(T)-g(T,W)r(Z)+\mathrm{Ric}(\overline{Z},W)\overline{T}-\mathrm{Ric}(\overline{T},W)\overline{Z}+\\&+g(\overline{Z},W)r(\overline{T})-g(\overline{T},W)r(\overline{Z})+2\,\mathrm{Ric}(\overline{Z},T)\overline{W}+2g(\overline{Z},T)r(\overline{W})]-\\&-\frac{R}{4(n+1)(n+2)}[g(Z,W)T-g(T,W)Z+g(\overline{Z},W)\overline{T}-g(\overline{T},W)\overline{Z}+2g(\overline{Z},T)\overline{W}].\end{aligned}$$

Hence, if a Kähler manifold is a Ricci birecurrent and a Bochner birecurrent manifold for the same recurrence parameter, it is also a projective birecurrent manifold.

Theorem 2.3. *We have*

$$\begin{aligned}(2.9)\quad B(Z,T,W)&=V(Z,T,W)+\frac{5}{2(n+2)(2n-1)}[\mathrm{Ric}(T,W)Z-\mathrm{Ric}(Z,W)T+\\&+r(Z)g(T,W)-r(T)g(Z,W)]+\frac{1}{2(n+2)}[\mathrm{Ric}(\overline{Z},W)\overline{T}-\mathrm{Ric}(\overline{T},W)\overline{Z}+\\&+g(\overline{Z},W)r(\overline{T})-g(\overline{T},W)r(\overline{Z})+2\,\mathrm{Ric}(\overline{Z},T)\overline{W}+2g(\overline{Z},T)r(\overline{W})-\\&-\frac{3R(3n-1)}{4(2n-1)(n^2-1)(n+2)}[g(T,W)Z-g(Z,W)T]-\\&-\frac{R}{4(n+1)(n+2)}[g(\overline{Z},W)\overline{T}-g(\overline{T},W)\overline{Z}+2g(\overline{Z},T)\overline{W}].\end{aligned}$$

Hence, if a Kähler manifold is a Ricci birecurrent and a Bochner birecurrent manifold for the same recurrence parameter, it is also a conformal birecurrent manifold.

Theorem 2.4. *We have*

$$\begin{aligned}(2.10)\quad B(Z,T,W)&=L(Z,T,W)+\frac{3}{2(n+2)(n-1)}[\mathrm{Ric}(T,W)Z-\mathrm{Ric}(Z,W)T+\\&+r(Z)g(T,W)-r(T)g(Z,W)]+\frac{1}{2(n+2)}[\mathrm{Ric}(\overline{Z},W)\overline{T}-\mathrm{Ric}(\overline{T},W)\overline{Z}+\\&+g(\overline{Z},W)r(\overline{T})-g(\overline{T},W)r(\overline{Z})+2\,\mathrm{Ric}(\overline{Z},T)\overline{W}+2g(\overline{Z},T)r(\overline{W})]-\\&-\frac{R}{4(n+1)(n+2)}[g(Z,W)T-g(T,W)Z+g(\overline{Z},W)\overline{T}-g(\overline{T},W)\overline{Z}+2g(\overline{Z},T)\overline{W}].\end{aligned}$$

Hence, if a Kähler manifold is a Ricci birecurrent and a Bochner birecurrent manifold for the same recurrence parameter it is also a conharmonic birecurrent manifold.

Theorem 2.5. *We have*

$$(2.11)\quad B(Z,T,W)=C(Z,T,W)+\frac{1}{2(n+2)}[\mathrm{Ric}(Z,W)T-\mathrm{Ric}(T,W)Z+ \\ +g(Z,W)r(T)-g(T,W)r(Z)+\mathrm{Ric}(\bar Z,W)\bar T-\mathrm{Ric}(\bar T,W)\bar Z+ \\ +g(\bar Z,W)r(\bar T)-g(\bar T,W)r(\bar Z)+2\,\mathrm{Ric}(\bar Z,T)+2g(\bar Z,T)r(\bar W)]+ \\ +\frac{(4n^2+5n+4)R}{4n(2n-1)(n+1)(n+2)}[g(T,W)Z-g(Z,W)T]- \\ -\frac{R}{4(n+1)(n+2)}[g(\bar Z,W)\bar T-g(\bar T,W)\bar Z+2g(\bar Z,T)\bar W].$$

Hence, if a Kähler manifold is a Ricci birecurrent manifold and a Bochner birecurrent manifold for the same recurrence parameter it is also a concircular birecurrent manifold.

Theorem 2.6. *We have*

$$(2.12)\quad B(Z,T,W)=P(Z,T,W)+\frac{1}{2(n+1)(n+2)}[\mathrm{Ric}(T,W)Z-\mathrm{Ric}(Z,W)T+ \\ +\mathrm{Ric}(\bar T,W)\bar Z+\mathrm{Ric}(Z,\bar W)\bar T+2\,\mathrm{Ric}(Z,\bar T)\bar W]+\frac{1}{2(n+2)}[g(Z,W)r(T)- \\ -g(T,W)r(Z)+g(\bar Z,W)r(\bar T)-g(\bar T,W)r(\bar Z)+2g(\bar Z,T)r(\bar W)]- \\ -\frac{R}{4(n+1)(n+2)}[g(Z,W)T-g(T,W)Z+g(\bar Z,W)\bar T-g(\bar T,W)\bar Z+2g(\bar Z,T)\bar W].$$

Hence, if a Kähler manifold is a Ricci birecurrent manifold and a Bochner birecurrent manifold for the sam recurrence parameter, it is also a H-projective birecurrent manifold.

Theorem 2.7. *We have*

$$(2.13)\quad B(Z,T,W)=T^*(Z,T,W)+\frac{1}{2(n+2)}[\mathrm{Ric}(Z,W)T-\mathrm{Ric}(T,W)Z+ \\ +g(Z,W)r(T)-g(T,W)r(Z)+\mathrm{Ric}(\bar Z,W)\bar T-\mathrm{Ric}(\bar T,W)\bar Z+g(\bar Z,W)r(\bar T)- \\ -g(\bar T,W)r(\bar Z)+2\,\mathrm{Ric}(\bar Z,T)\bar W+2g(\bar Z,T)r(\bar W)]-\frac{R}{2n(n+2)}[g(Z,W)T- \\ -g(T,W)Z+g(\bar Z,W)\bar T-g(\bar T,W)\bar Z+g(\bar Z,T)\bar W]-\frac{R}{4(n+1)(n+2)}[g(\bar Z,T)\bar W].$$

Hence, if a Kähler manifold is a Ricci birecurrent manifold and a Bochner birecurrent manifold for the same recurrence parameter, it is also a T-concircular birecurrent manifold.

3. *Holomorphic sectional curvature:*

Theorem (3.1). *The holomorphic sectional curvature k of a Kähler manifold M_{2n} with respect to a vector X is given by, in terms of Bochner curvature tensor,*

$$k = -\frac{'B(X, \bar{X}, X, \bar{X})}{g(X, X)g(X, X)} + \frac{4\,\mathrm{Ric}\,(X, X)}{(n+2)g(X, X)} - \frac{R}{(n+1)(n+2)}. \tag{3.1}$$

PROOF. By putting $Z=X$, $T=\bar{X}$, $W=X$ and $U=\bar{X}$ in (2.3) and using (1.1), (1.3) and (1.4), we get

(3.2)

$$'B(X, \bar{X}, X, \bar{X}) = 'K(X, \bar{X}, X, \bar{X}) + \frac{4}{(n+2)}\, g(X, X)\,\mathrm{Ric}\,(X, X) - \frac{Rg(X, X)g(X, X)}{(n+1)(n+2)}$$

Substituting $'K(X, \bar{X}, X, \bar{X})$ from (1.15) in (3.2), we get

$$'B(X, \bar{X}, X, \bar{X}) = \tag{3.3}$$

$$= -kg(X, X)g(X, X) + \frac{4}{(n+2)}\, g(X, X)\,\mathrm{Ric}\,(X, X) - \frac{Rg(X, X)g(X, X)}{(n+1)(n+2)}$$

which proves the statement.

Corollary 3.1. The holomorphic sectional curvature k in the direction of the unit vector X is given by

$$k = -'B(X, \bar{X}, X, \bar{X}) + \frac{4}{n+2}\,\mathrm{Ric}\,(X, X) - \frac{R}{(n+1)(n+2)}. \tag{3.4}$$

PROOF. We have $g(X, X)=1$.
Using it in (3.1), we get (3.4).

Theorem 3.2. *For a Kähler manifold of constant curvature, we have*

$$'B(Z, T, W, U) = \frac{3k}{2(n+1)}\,[g(Z, U)g(T, W) - g(Z, W)g(T, U)] + \tag{3.5}$$

$$+ \frac{(2n-1)k}{(n+1)}\,[g(\bar{Z}, W)g(\bar{T}, U) - g(\bar{Z}, U)g(\bar{T}, W) + 2g(\bar{Z}, T)g(\bar{W}, U)].$$

PROOF. It the Kähler manifold is of constant curvature k, then we know that

$$'K(Z, T, W, U) = k[g(Z, U)g(T, W) - g(Z, W)g(T, U)], \tag{3.6}$$

$$K(Z, T, W) = k[g(T, W)Z - g(Z, W)T], \tag{3.7}$$

$$\mathrm{Ric}\,(T, W) = k[2ng(T, W) - g(T, W)], \tag{3.8}$$

and

$$r(T) = (2n-1)kT. \tag{3.9}$$

Contracting (3.9), we have

$$R = 2n(2n-1)k. \tag{3.10}$$

Using (3.6), (3.8) and (3.10), we get (3.5).

Corollary (3.2). If a Kähler manifold of constant curvature is flat, then Bochner curvature tensor vanishes.

PROOF. By putting $k=0$ in (3.5), we get

$$'B(Z, T, W, U) = 0.$$

Hence the statement.

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