On the convolution of distributions

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1. *Introduction.*

The definition of the convolution of two distributions was extended recently in a paper by JONES [2] to cover certain pairs of distributions which could not be convolved in the sense of the previous definition. The convolution $f*g$ of two distributions f and g was defined as the limit of the sequence $\{f_n*g_n\}$, provided the limit h exists in the sense that

$$\lim_{n\to\infty}\int \varphi(x)(f_n*g_n)(x)\,dx=\int \varphi(x)h(x)\,dx$$

for all fine functions φ in the terminology of JONES [3], where

$$f_n(x)=f(x)\tau(x/n),\quad g_n(x)=g(x)\tau(x/n)$$

and τ is an infinitely differentiable function satisfying the following conditions:

(1) $$\tau(x)=\tau(-x),$$

(2) $$0\leqq\tau(x)\leqq 1,$$

(3) $$\tau(x)=1\quad\text{for}\quad |x|\leqq\frac{1}{2},$$

(4) $$\tau(x)=0\quad\text{for}\quad |x|\geqq 1.$$

It was proved in [2] that with this definition

$$1*\operatorname{sgn} x=x$$

and in [1] the generalization

(1) $$x^r*(\operatorname{sgn} x\cdot x^r)=\frac{(r!)}{(2r+1)!}\,x^{2r+1}$$

was proved, for $r=0,1,2,\ldots$.

In the following we prove a theorem which gives equation (1) as a particular case.

2. Theorem. *Let f and g be arbitrary distributions for which*

$$f(x)=0\quad\textit{for}\quad x\leqq a,$$

$$g(x)=0\quad\textit{for}\quad x\geqq b,$$

where a, b are arbitrary real numbers. Then the convolution $(f+g)*(f-g)$ *exists and*

$$(f+g)*(f-g) = f*f-g*g.$$

PROOF. We note first of all that the convolutions $f*f$ and $g*g$ exist without the use of Jones' definition of the convolution.

We write

$$(f+g)_n(x) = (f+g)(x)\tau(x/n),$$

$$(f-g)_n(x) = (f-g)(x)\tau(x/n).$$

It follows that the convolution $(f+g)_n*(f-g)_n$ exists and

$$(f+g)_n*(f-g)_n = (f_n+g_n)*(f_n-g_n) = f_n*f_n+g_n*f_n-f_n*g_n-g_n*g_n.$$

Since

$$g_n*f_n = f_n*g_n$$

and

$$\lim_{n\to\infty} f_n*f_n = f*f, \quad \lim_{n\to\infty} g_n*g_n = g*g$$

it follows that

$$\lim_{n\to\infty} (f+g)_n*(f-g)_n = f*f-g*g,$$

completing the proof of the theorem.

An obvious extension of this theorem is that if h_1 and h_2 are arbitrary distributions with compact support and f and g are distributions satisfying the conditions of the theorem, then the convolution $(f+g+h_1)*(f-g+h_2)$ exists and

$$(f+g+h_1)*(f-g+h_2) = f*f-g*g+h_1*(f-g)+h_2*(f+g),$$

the convolutions $h_1*(f-g)$ and $h_2*(f+g)$ existing since h_1 and h_2 have compact support.

Since

$$x_+^\alpha * x_+^\beta = B(\alpha+1, \beta+1)x_+^{\alpha+\beta+1},$$

$$x_-^\alpha * x_-^\beta = B(\alpha+1, \beta+1)x_-^{\alpha+\beta+1},$$

for $\alpha, \beta, \alpha+\beta+1 \neq -1, -2, \dots,$ where B denotes the beta function, it follows from the theorem that

$$(x_+^\alpha + x_-^\beta)*(x_+^\alpha - x_-^\beta) = B(\alpha+1, \alpha+1)x_+^{2\alpha+1} - B(\beta+1, \beta+1)x_-^{2\beta+1}$$

for

$$\alpha, \beta \neq -1, -2, \dots,$$

and $(x_+^\alpha + x_-^\alpha)*(x_+^\alpha - x_-^\alpha) = |x|^\alpha*(\operatorname{sgn} x\cdot|x|^\alpha) = B(\alpha+1, \alpha+1)\operatorname{sgn} x\cdot|x|^{2\alpha+1}$

for $\alpha \neq -1, -2, \dots,$ the particular case $\alpha=r=0, 1, 2, \dots,$ being equation (1).

More generally it follows that the convolutions

$$(x_+^\alpha \ln^p x_+ + x_-^\beta \ln^q x_-)*(x_+^\alpha \ln^p x_- - x_-^\beta \ln^q x_-)$$

and

$$(|x|^\alpha \ln^p |x|)*(\operatorname{sgn} x\cdot|x|^\alpha \ln^p |x|)$$

exist for all α, β and $p, q=0, 1, 2, \dots.$

References

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[3] D. S. JONES, Generalized functions, *New York,* 1966.

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