# On effectiveness of basic sets of polynomials associated with functions of algebraic infinite matrices

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This note involves generalizations of some of the results in [1] and [2]. We assume that the reader is familiar with [6].

A matrix function $F(P)$ of an algebraic infinite matrix $P$ of degree $m$, can be expressed as a polynomial of degree at most $m-1$ in $P$ [1]). If $P$ is row-finite and $F(P)$ is non-singular we can associate with $Q=F(P)$ a basic set of ploynomials $\{q_n(z)\}$. Writing $Q=[q_{ni}]$, $n, i=0, 1, 2, \ldots, q_n(z)$ is defined by

$$q_n(z) = \sum_i q_{ni} z^i. \tag{1}$$

The matrix $Q$ has a unique row-finite reciprocal $W=[w_{ni}]$ where

$$z^n = \sum_i w_{ni} q_i(z). \tag{2}$$

We prove the following result.

**Theorem 1.** *Let $P$ be an algebraic row-finite matrix having the minimum equation*

$$f(P) = \prod_{i=1}^{t} (P-u_i I)^{m_i} = 0, \quad \sum_{i=1}^{t} m_i = m. \tag{3}$$

*If the basic set of polynomials associated with any non-singular matrix function $Q=F(P)$ satisfying the conditions*

$$F(u_i) \neq F(u_j), \quad \text{whenever} \quad i \neq j \tag{4}$$

$$F'(u_i) \neq 0, \quad \text{whenever} \quad m_i > 1 \tag{5}$$

$$\varliminf_{n\to\infty} |w_{nn}|^{1/n} = \delta \geqq 1 \tag{6}$$

*is effective in $|z| \leqq R$ for $a \leqq R < b$, then so also is the basic set associated with any non-singular matrix function $\Theta(P)$.*

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[1]) For the definition of a matrix function $F(P)$ and the reduction of $F(P)$ to a polynomial of degree at most $m-1$ in $P$, it is sufficient to refer to [3, pp. 208—209].

The proof will make use of a generalization of theorem 1 of [1]. Writing $P=[p_{ni}]$, $p_n(z)=\sum_i p_{ni} z^i$, and

$$A(R) = \overline{\lim_{n\to\infty}} \{A_n(R)\}^{1/n} = \overline{\lim_{n\to\infty}} \{\max_{|z|=R} |p_n(z)|\}^{1/n} \tag{7}$$

this generalization is:

**Lemma.** *If an algebraic row-finite matrix $P$ is such that $A(R) \leqq R$ for $a \leqq R < b$, then the basic set $\{q_n(z)\}$ associated with any non-singular matrix function $Q=F(P)$ is effective in $|z| \leqq R$ for $a \leqq R < b$.*

For writing

$$B(R) = \overline{\lim_{n\to\infty}} \{B_n(R)\}^{1/n} = \overline{\lim_{n\to\infty}} \{\max_{|z|=R} |q_n(z)|\}^{1/n} \tag{8}$$

and applying the result in lemma 1 of [1] to the matrix

$$Q = a_0 I + a_1 P + \ldots + a_{m-1} P^{m-1} \tag{9}$$

we get $B(R) \leqq R$ for $a \leqq R < b$. The matrix $Q$, being a polynomial in an algebraic matrix $P$, is itself an algebraic matrix [5, pp. 433—434]. Applying th. 1 of [1] to the algebraic matrix $Q$ we get the result in the above lemma [2]).

Now, we turn to prove the theorem.

Writing

$$T_n(R) = \max_{i,j} \max_{|z|=R} |w_{ni} q_i(z) + w_{n,i+1} q_{i+1}(z) + \ldots + w_{nj} q_j(z)| \tag{10}$$

$$\Lambda(R) = \overline{\lim_{n\to\infty}} \{T_n(R)\}^{1/n}, \tag{11}$$

we have by th. 24 of [6],

$$\Lambda(R) = R, \quad \text{for} \quad a \leqq R < b. \tag{12}$$

From the fact that

$$|w_{nn}| B_n(R) \leqq T_n(R), \tag{13}$$

from the condition in (6) and from (12), we deduce that

$$B(R) \leqq R, \quad \text{for} \quad a \leqq R < b. \tag{14}$$

Now, since $F(z)$ satisfies the conditions in (4) and (5), $P$ can be expressed as a polynomial in $Q$ [3]). Since the matrix function $\Theta(P)$ can be expressed as a polynomial in $P$, it can also be expressed as a polynomial in $Q$. Since $Q$ satisfies (14) then by the above lemma the basic set associated with $\Theta(P)$, provided $\Theta(P)$ is non-singular, is effective in $|z| \leqq R$, for $a \leqq R < b$.

Similarly, applying the generalization of th. 2 of [1] we can prove that:

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[2]) The authors have given a (lengthy unpublished) proof of the lemma without making use of th. 1 of [1]; the above simple proof is due to S. A. SHEHATA.

[3]) Indeed (4) and (5) are necessary and sufficient conditions that $P$ should be written as a polynomial in $Q=F(P)$, [4, th. III].

**Theorem 2.** *With the conditions in* (4), (5) *and* (6), *if the basic set associated with* $F(P)$ *is effective in the open circle* $|z|<R$ *then so also is the basic set associated with any* $\Theta(P)$.

Applying the generalization of th. 3 of [1], we can prove that:

**Theorem 3.** *With the conditions in* (4), (5) *and*

$$\varliminf_{n\to\infty} |w_{nn}|^{1/n} = \delta > 0 \tag{15}$$

*if the basic set associated with* $F(P)$ *is effective at the origin then so also is the basic set associated with any* $\Theta(P)$.

Here, we have $\wedge(0+)=0$, [6, th. 30] and the condition in (15) together with (13) gives $B(R)\leqq\wedge(R)/\delta$, so that making $R\to 0+$, we get $B(0+)=0$, and the proof is completed as in theorem 1.

Also, applying the generalization of th. 4 of [1], we have

**Theorem 4.** *If the matrix* $P$ *of theorem* 1 *satisfies*

$$\varlimsup_{n\to\infty} \frac{u(n)}{n} < \infty \tag{16}$$

*where* $u(n)$ *is the degree of* $p_n(z)$, *then with the conditions in* (4), (5) *and* (15), *the effectiveness of the basic set associated with* $F(P)$ *for every entire function implies the same for the basic set associated with any* $\Theta(P)$.

From theorem 1 we deduce the following:

*Cor. 1:* Let $P$ be an algebraic lower semi-matrix having the minimum equation in (3). Then if the basic set associated with any non-singular matrix function $F(P)$ satisfying the conditions in (4) and (5) is effective in $|z|\leqq R$, so also is the basic set associated with any non-singular matrix function $\Theta(P)$.

Since $P$ is a lower semi-matrix its diagonal elements are the numbers $u_i$, $i=1,2,\dots,t$ distributed in some way. The matrix $Q=F(P)$ is then a lower semi-matrix and its diagonal elements are the numbers $F(u_i)$ distributed in the same way. Since $F(P)$ is a non-singular matrix $F(u_i)\neq 0$, $i=1,2,\dots,t$. The matrix $W=Q^{-1}$ is a lower semi-matrix with the diagonal elements $1/F(u_i)$. If we write $\alpha$ and $\beta$ respectively for the maximum and minimum of $|F(u_i)|$, $i=1,2,\dots,t$, then

$$\frac{1}{\alpha} \leqq |w_{nn}| \leqq \frac{1}{\beta}, \quad \text{for all} \quad n.$$

Hence

$$|w_{nn}|^{1/n} \to 1 \quad \text{as} \quad n\to\infty. \tag{17}$$

Thus condition (6) of theorem 1 is satisfied. Since the basic set associated with $F(P)$ is a simple set, then if it is effective in $|z|\leqq R$, it is effective in $|z|\leqq\varrho$ for all $\varrho\geqq R$ [6, th. 12], and by theorem 1 so also is the (simple) basic set associated with any non-singular matrix function $\Theta(P)$.

Theorem 2 does not give a result independent of that in cor. 1, for a simple set which is effective in an open circle $|z|<R$ is also effective in the closed circle $|z|\leqq R$.

From theorems 3 and 4 we have resptecively:

*Cor. 2:* With the conditions of cor. 1, the effectiveness at the origin of the basic set associated with $F(P)$ implies the same for the basic set associated with any $\Theta(P)$.

*Cor. 3:* With the conditions of cor. 1, the effectiveness of the basic set associated with $F(P)$ for every entire function implies the same for the basic set associated with any $\Theta(P)$.

Here we have $u(n)=n$ for all $n$, so that condition (16) is satisfied.

The results in corollaries 1, 2 and 3 are refined general forms of those in § 1 of [2].

## References

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