Some problems on horizontal and complete lifts of $\Theta(4, -2)$-structure

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Summary.

The horizontal and complete lifts from a differentiable manifold M of class c^∞ to its cotangent bundle $cT(M)$ have been studied by YANO and PATTERSON ([2], [3]). YANO, HOUH and CHEN [4] have studied the structures defined by a tensor field Θ of type (1, 1) satisfying $\Theta^4 \pm \Theta^2 = 0$. The purpose of the present paper is to obtain certain results on horizontal and complete lifts of $\Theta(4, -2)$-structure.

In section 2, we have obtained the conditions under which the complete lift of a tensor field Θ of type (1, 1) having a $\Theta(4, -2)$-structure in M may have a $\Theta(4, -2)$-structure in $cT(M)$. In section 3, we have proved that the processes of computing the Nijenhuis tensor of Θ^2 and taking the complete lift are commutative. In section 4, we have proved that the horizontal lift of a tensor field Θ of type (1, 1) having a $\Theta(4, -2)$-structure in M will also have a $\Theta(4, -2)$-structure in $cT(M)$.

1. *Preliminaries.*

Let M be a differentiable manifold of class c^∞ and dimension n, and let $cT(M)$ be the cotangent bundle of M. Then $cT(M)$ is also a differentiable manifold of class c^∞; the dimension of $cT(M)$ is $2n$.

Throughout this paper, we use the following notations and conventions:

(i) $\pi : cT(M) \to M$ is the projection of $cT(M)$ onto M.

(ii) Suffixes $a, b, c, \dots, h, i, j, \dots$ take the values 1 to n and $\bar{i} = i + n$, etc. Suffixes $A, B, C, D \dots$ take the values 1 to $2n$.

(iii) $\mathscr{I}^r_s(M)$ denotes the set of tensor fields of class c^∞ and type (r, s) in M, and $\mathscr{I}^r_s(cT(M))$ denotes the corresponding set of tensor fields in $cT(M)$.

(iv) Vector fields in M are denoted by X, Y, Z. The Lie product of X and Y is denoted by $[X, Y]$ and the Lie derivative with respect to X is denoted by $\mathscr{L}_X$.

If A is a point in M, then $\pi^{-1}(A)$ is the fibre over A. Any point $P \in \pi^{-1}(A)$ is an ordered pair (A, p_A), where p is a 1-form in M and p_A is its value at A. Let U be a coordinate neighbourhood in M such that $A \in U$. Then U induces a coordinate neighbourhood $\pi^{-1}(U)$ in $cT(M)$ and $P \in \Pi^{-1}(U)$.

If $X, Y, Z \in \mathscr{I}^1_0(M)$ and $\Theta \in \mathscr{I}^1_1(M)$. Then we have [2]

$$(1.1) \qquad (X+Y)^C = X^C + Y^C,$$

$$(1.2) \qquad \Theta^C(Z^C) = (\Theta Z)^C + (\mathscr{L}_Z \Theta)^V,$$

where X^C, Y^C, Z^C are the complete lifts of the vector fields X, Y, Z respectively and Θ^C is the complete lift of the tensor field Θ.

2. *The complete lift of a $\Theta(4, -2)$-structure.*

Let Θ be a tensor field of type (1, 1) and of class c^∞ in M which implies that $\Theta \in \mathscr{I}_1^1(M)$ and let Θ satisfy the following:

$$\Theta^4 - \Theta^2 = 0. \tag{2.1}$$

That is, Θ is a $\Theta(4, -2)$-structure on M [4].

Let Θ_i^h be the components of Θ at a point A in a coordinate neighbourhood U. Then the complete lift Θ^C of Θ is also a tensor field of type (1, 1) in $cT(M)$ whose components $\tilde{\Theta}_B^A$ in $\pi^{-1}(U)$ are given by [2]

$$\begin{aligned} &\tilde{\Theta}_i^h = \Theta_i^h, \quad \tilde{\Theta}_{\bar{i}}^h = 0, \\ &\tilde{\Theta}_i^{\bar{h}} = p_a\left(\frac{\partial \Theta_h^a}{\partial x^i} - \frac{\partial \Theta_i^a}{\partial x^h}\right), \quad \tilde{\Theta}_{\bar{i}}^{\bar{h}} = \Theta_h^i, \end{aligned} \tag{2.2}$$

where $(x^1, x^2, \dots x^n)$ are coordinates of A relative to U and p_A has components $(p_1, p_2, \dots, p_n)$. Therefore, we have

$$\Theta^C \overset{\text{def}}{=} \tilde{\tilde{\Theta}}_i^h = \begin{pmatrix} \Theta_i^h & 0 \\ p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a) & \Theta_h^i \end{pmatrix}, \tag{2.3}$$

where $\partial_i \equiv \partial / \partial x^i$.

Now, in consequence of (2.3), we have

$$\begin{aligned} \tilde{\tilde{\Theta}}_i^h \tilde{\tilde{\Theta}}_j^i &= \begin{pmatrix} \Theta_i^h & 0 \\ p_a(\partial_i \Theta_h^a - \partial_h \Theta_i^a) & \Theta_h^i \end{pmatrix} \begin{pmatrix} \Theta_j^i & 0 \\ p_t(\partial_j \Theta_i^t - \partial_i \Theta_j^t) & \Theta_i^j \end{pmatrix}, \\ &= \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ 2\Theta_j^i p_a \partial_{[i} \Theta_{h]}^a + 2\Theta_h^i p_t \partial_{[j} \Theta_{i]}^t & \Theta_h^i \Theta_i^j \end{pmatrix}, \end{aligned} \tag{2.4}$$

where $2\partial_{[i}\Theta_{h]}^a \overset{\text{def}}{=} (\partial_i \Theta_h^a - \partial_h \Theta_i^a)$.

Let us put

$$L_{hj} \overset{\text{def}}{=} 2(\Theta_j^i p_a \partial_{[i} \Theta_{h]}^a + \Theta_h^i p_t \partial_{[j} \Theta_{i]}^t). \tag{2.5}$$

Then in view of (2.5), equation (2.4) becomes

$$\tilde{\tilde{\Theta}}_i^h \tilde{\tilde{\Theta}}_j^i = \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ L_{hj} & \Theta_h^i \Theta_i^j \end{pmatrix}. \tag{2.6}$$

Hence we have

$$\begin{aligned} \tilde{\tilde{\Theta}}_i^h \tilde{\tilde{\Theta}}_j^i \tilde{\tilde{\Theta}}_k^j \tilde{\tilde{\Theta}}_l^k &= \begin{pmatrix} \Theta_i^h \Theta_j^i & 0 \\ L_{hj} & \Theta_h^i \Theta_i^j \end{pmatrix} \begin{pmatrix} \Theta_k^j \Theta_l^k & 0 \\ L_{jl} & \Theta_j^k \Theta_k^l \end{pmatrix}, \\ &= \begin{pmatrix} \Theta_i^h \Theta_j^i \Theta_k^j \Theta_l^k & 0 \\ L_{hj} \Theta_k^j \Theta_l^k + \Theta_h^i \Theta_i^j L_{jl} & \Theta_h^i \Theta_i^j \Theta_j^k \Theta_k^l \end{pmatrix}. \end{aligned} \tag{2.7}$$

Now we shall prove the following lemma and theorems:

Lemma (2.1). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$-structure in M may have $\Theta(4, -2)$-structure in $cT(M)$, it is necessary and sufficient that*

$$L_{hj}\,\Theta^j_k\,\Theta^k_l + \Theta^i_h\,\Theta^j_i\,L_{jl} = L_{hl}. \tag{2.8}$$

PROOF. Since Θ satisfies $\Theta^4 - \Theta^2 = 0$, therefore

$$\Theta^h_i\,\Theta^i_j\,\Theta^j_k\,\Theta^k_l = \Theta^h_r\,\Theta^r_l. \tag{2.9}$$

Thus in view of (2.7) and (2.9), we obtain

$$\tilde{\tilde{\Theta}}{}^h_i\,\tilde{\tilde{\Theta}}{}^i_j\,\tilde{\tilde{\Theta}}{}^j_k\,\tilde{\tilde{\Theta}}{}^k_l = \begin{pmatrix} \Theta^h_r\Theta^r_l & 0 \\ L_{hj}\,\Theta^j_k\,\Theta^k_l + \Theta^i_h\,\Theta^j_i\,L_{jl} & \Theta^r_h\Theta^l_r \end{pmatrix}. \tag{2.10}$$

Let us assume that the condition (2.8) is satisfied. Then from (2.10) we have

$$\tilde{\tilde{\Theta}}{}^h_i\,\tilde{\tilde{\Theta}}{}^i_j\,\tilde{\tilde{\Theta}}{}^j_k\,\tilde{\tilde{\Theta}}{}^k_l = \begin{pmatrix} \Theta^h_r\Theta^r_l & 0 \\ L_{hl} & \Theta^r_h\Theta^l_r \end{pmatrix},$$

which in view of (2.6) yields

$$\tilde{\tilde{\Theta}}{}^h_i\,\tilde{\tilde{\Theta}}{}^i_j\,\tilde{\tilde{\Theta}}{}^j_k\,\tilde{\tilde{\Theta}}{}^k_l = \tilde{\tilde{\Theta}}{}^h_r\,\tilde{\tilde{\Theta}}{}^r_l.$$

Thus Θ^C in $cT(M)$ satisfies

$$(\Theta^C)^4 - (\Theta^C)^2 = 0.$$

The necessary condition can be proved in a straight forward manner.

Theorem (2.1). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$-structure in M may have $\Theta(4, -2)$-structure in $cT(M)$, it is necessary and sufficient that*

$$\begin{aligned} &\Theta^t_j\,\Theta^j_k\,\Theta^k_l\,p_a\,\partial_{[i}\Theta^a_{h]} + \Theta^i_h\,\Theta^j_k\,\Theta^k_l\,p_a\,\partial_{[j}\Theta^a_{i]} + \Theta^i_h\,\Theta^j_i\,\Theta^s_l\,p_r\,\partial_{[s}\Theta^r_{j]} + \\ &+ \Theta^i_h\,\Theta^j_i\,\Theta^s_j\,p_r\,\partial_{[jl}\Theta^r_{s]} - \Theta^t_l\,p_a\,\partial_{[t}\Theta^a_{h]} - \Theta^t_h\,p_a\,\partial_{[l}\Theta^a_{t]} = 0. \end{aligned} \tag{2.11}$$

PROOF. From lemma (2.1) it follows that the complete lift Θ^C in $cT(M)$ of Θ in M satisfying $\Theta^4 - \Theta^2 = 0$ will satisfy $(\Theta^C)^4 - (\Theta^C)^2 = 0$, if and only if (2.8) is satisfied. Now the condition (2.8) in view of (2.5) can be expressed as

$$\begin{aligned} &\{\Theta^i_j\,p_a\,\partial_{[i}\Theta^a_{h]} + \Theta^i_h\,p_a\,\partial_{[j}\Theta^a_{i]}\}\,\Theta^j_k\,\Theta^k_l + \Theta^i_h\,\Theta^j_i\{\Theta^s_l\,p_r\,\partial_{[s}\Theta^r_{j]} + \Theta^s_j\,p_r\,\partial_{[l}\Theta^r_{s]}\} = \\ &= \Theta^t_l\,p_a\,\partial_{[t}\Theta^a_{h]} + \Theta^t_h\,p_a\,\partial_{[l}\Theta^a_{t]}, \end{aligned}$$

or

$$\begin{aligned} &\Theta^i_j\,p_a\,\partial_{[i}\Theta^a_{h]}\,\Theta^j_k\,\Theta^k_l + \Theta^i_h\,p_a\,\partial_{[j}\Theta^a_{i]}\,\Theta^j_k\,\Theta^k_l + \Theta^i_h\,\Theta^j_i\,\Theta^s_l\,p_r\,\partial_{[s}\Theta^r_{j]} + \\ &+ \Theta^i_h\,\Theta^j_i\,\Theta^s_j\,p_r\,\partial_{[l}\Theta^r_{s]} - \Theta^t_l\,p_a\,\partial_{[t}\Theta^a_{h]} - \Theta^t_h\,p_a\,\partial_{[l}\Theta^a_{t]} = 0. \end{aligned} \tag{2.12}$$

Thus the theorem follows.

Theorem (2.2). *In order that the complete lift Θ^C in $cT(M)$ of a tensor field Θ having $\Theta(4, -2)$-structure in M may have $\Theta(4, -2)$-structure in $cT(M)$, it is necessary and sufficient that*

$$(\operatorname{curl} V + 2p_a\Gamma^a_{r[h}\Theta^r_{i]})^4 - (\operatorname{curl} V + 2p_a\Gamma^a_{r[h}\Theta^r_{i]})^2 = 0, \tag{2.13}$$

where we have defined the following:

$$\text{curl } V = V_{h,i} - V_{i,h} \overset{\text{def}}{=} p_a(\Theta^a_{h,i} - \Theta^a_{i,h}) \tag{2.14}$$

and

$$2\Gamma^a_{r[h}\Theta^r_{i]} \overset{\text{def}}{=} \Gamma^a_{rh}\Theta^r_i - \Gamma^a_{ri}\Theta^r_h. \tag{2.15}$$

PROOF. Let us assume that the complete lift Θ^C in $cT(M)$ of Θ having $\Theta(4, -2)$-structure in M has a $\Theta(4, -2)$-structure in $cT(M)$. Since in view of (2.2), the only relevant part of the components of Θ^C in $cT(M)$ is

$$p_a(\partial_i\Theta^a_h - \partial_h\Theta^a_i), \quad \text{where} \quad \partial_i \equiv \partial/\partial x^i.$$

Therefore Θ^C satisfies

$$[p_a(\partial_i\Theta^a_h - \partial_h\Theta^a_i)]^4 - [p_a(\partial_i\Theta^a_h - \partial_h\Theta^a_i)]^2 = 0. \tag{2.16}$$

Let Γ^i_{jk} be a symmetric affine connection in M. Then the covariant derivative of Θ^a_h with respect to x^i is given by

$$\Theta^a_{h,i} = \partial_i\Theta^a_h + \Gamma^a_{ri}\Theta^r_h - \Gamma^s_{hi}\Theta^a_s. \tag{2.17}$$

In consequence of (2.17), equation (2.16) assumes the form

$$[p_a(\Theta^a_{h,i} - \Theta^a_{i,h} + \Gamma^a_{rh}\Theta^r_i - \Gamma^a_{ri}\Theta^r_h)]^4 - [p_a(\Theta^a_{h,i} - \Theta^a_{i,h} + \Gamma^a_{rh}\Theta^r_i - \Gamma^a_{ri}\Theta^r_h)]^2 = 0.$$

As regards a fixed, Θ is covariant. Hence in view of (2.15) we have

$$[p_a\{(\Theta^a_{h,i} - \Theta^a_{i,h}) + 2\Gamma^a_{r[h}\Theta^r_{i]}\}]^4 - [p_a\{(\Theta^a_{h,i} - \Theta^a_{i,h}) + 2\Gamma^a_{r[h}\Theta^r_{i]}\}]^2 = 0. \tag{2.18}$$

Equation (2.18) in consequence of (2.14) yields (2.13). Similarly, it can also be proved that the condition is sufficient.

Corollary (2.1). In a Kähler space with $\Theta(4, -2)$-structure, the complete lift Θ^C of Θ has a $\Theta(4, -2)$-structure in $cT(M)$ if and only if

$$(2p_a\Gamma^a_{r[h}\Theta^r_{i]})^4 - (2p_a\Gamma^a_{r[h}\Theta^r_{i]})^2 = 0. \tag{2.19}$$

PROOF. For a Kählerian space, we have [1]

$$\Theta^a_{h,i} = 0 \quad \text{or} \quad \text{curl } V = 0. \tag{2.20}$$

Thus by virtue of (2.13) and (2.20), the result follows.

Corollary (2.2). In a manifold M with $\Theta(4, -2)$-structure if curl $V=0$, then Θ^C has a $\Theta(4, -2)$-structure in $cT(M)$ if and only if (2.19) is satisfied.

PROOF. The proof is similar to that of Corollary (2.1).

3. *Nijenhuis tensor of the complete lift of* Θ^4.

Let $\Theta \in \mathscr{I}^1_1(M)$. Then the Nijenhuis tensor of Θ is given by [2]

$$N_{\Theta,\Theta}(X, Y) = [\Theta X, \Theta Y] - \Theta[\Theta X, Y] - \Theta[X, \Theta Y] + \Theta^2[X, Y]. \tag{3.1}$$

For every $\Theta \in \mathscr{I}^1_1(M)$, we also have [2]

$$(\Theta^C)^2 = (\Theta^2)^C + (N_{\Theta,\Theta})^V. \tag{3.2}$$

Let Θ be a $\Theta(4, -2)$-structure on M. That is, $\Theta^4 - \Theta^2 = 0$. Now we shall prove the following theorems:

Theorem (3.1). *The Nijenhuis tensor of the complete lift of Θ^4 vanishes if Lie derivatives of the tensor field Θ^2 with respect to X and Y are both zero and Θ is an almost product structure on M.*

PROOF. In consequence of (3.1), the Nijenhuis tensor of $(\Theta^4)^C$ is given by

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = [(\Theta^4)^C X^C, (\Theta^4)^C Y^C] - (\Theta^4)^C[(\Theta^4)^C X^C, Y^C] - -(\Theta^4)^C[X^C, (\Theta^4)^C Y^C] + (\Theta^4)^C(\Theta^4)^C[X^C, Y^C],$$

which in view of (2.1) yields

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = [(\Theta^2)^C X^C, (\Theta^2)^C Y^C] - -(\Theta^2)^C[(\Theta^2)^C X^C, Y^C] - (\Theta^2)^C[X^C, (\Theta^2)^C Y^C] + (\Theta^2)^C(\Theta^2)^C[X^C, Y^C]. \tag{3.3}$$

In consequence of (1.2), we have

$$(\Theta^2)^C X^C = (\Theta^2 X)^C + (\mathcal{L}_X \Theta^2)^V. \tag{3.4}$$

Therefore by virtue of (3.4), equation (3.3) becomes

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = [(\Theta^2 X)^C, (\Theta^2 Y)^C] + [(\mathcal{L}_X \Theta^2)^V, (\Theta^2 Y)^C] + +[(\Theta^2 X)^C, (\mathcal{L}_Y \Theta^2)^V] + [(\mathcal{L}_X \Theta^2)^V, (\mathcal{L}_Y \Theta^2)^V] - (\Theta^2)^C[(\Theta^2 X)^C, Y^C] - -(\Theta^2)^C[(\mathcal{L}_X \Theta^2)^V, Y^C] - (\Theta^2)^C[X^C, (\Theta^2 X)^C] - -(\Theta^2)^C[X^C, (\mathcal{L}_Y \Theta^2)^V] + (\Theta^2)^C(\Theta^2)^C[X^C, Y^C]. \tag{3.5}$$

If Lie derivatives of the tensor field Θ^2 with respect to X and Y are both zero. Then we have

$$\mathcal{L}_X \Theta^2 = 0 \quad \text{and} \quad \mathcal{L}_Y \Theta^2 = 0.$$

Therefore equation (3.5) reduces to

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = [(\Theta^2 X)^C, (\Theta^2 Y)^C] - -(\Theta^2)^C[(\Theta^2 X)^C, Y^C] - (\Theta^2)^C[X^C, (\Theta^2 Y)^C] + (\Theta^2)^c(\Theta^2)^C[X^c, Y^C]. \tag{3.6}$$

Now for every $X, Y \in \mathcal{I}_0^1(M)$, we have [2]

$$[X^C, Y^C] = [X, Y]^C \tag{3.7}$$

Therefore by virtue of (3.7), equation (3.6) becomes

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = [\Theta^2 X, \Theta^2 Y]^C - (\Theta^2)^C[\Theta^2 X, Y]^C - -(\Theta^2)^C[X, \Theta^2 Y]^C + (\Theta^2)^C(\Theta^2)^C[X, Y]^C. \tag{3.8}$$

If Θ is also an almost product structure on M, $\Theta^2 = I$, where I is the unit tensor field. Hence, equation (3.8) yields

$$N_{(\Theta^4)^C,(\Theta^4)^C}(X^C, Y^C) = 0.$$

Theorem (3.2). *The Nijenhuis tensor of the complete lift of Θ^4 is equal to the complete lift of the Nijenhuis tensor of Θ^2 if*

(i) $$\mathscr{L}_X \Theta^2 = 0, \quad \mathscr{L}_Y \Theta^2 = 0$$

and

(ii) $$[X, Y]^C = 0, \quad \tilde{K}^V = 0;$$

where

$$\tilde{K} \overset{\text{def}}{=} \mathscr{L}_{[\Theta^2 X, Y]} \Theta^2 + \mathscr{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathscr{L}_{[X, Y]} \Theta^4. \tag{3.9}$$

Proof. In consequence of (1.1) and (3.1), we have

$$(N_{\Theta^2, \Theta^2}(X, Y))^C = [\Theta^2 X, \Theta^2 Y]^C - (\Theta^2[\Theta^2 X, Y])^C - (\Theta^2[X, \Theta^2 Y])^C + (\Theta^4[X, Y])^C,$$

which in view of (3.4) yields

$$\begin{aligned}(N_{\Theta^2, \Theta^2}(X, Y))^C = [\Theta^2 X, \Theta^2 Y]^C - \{(\Theta^2)^C[\Theta^2 X, Y]^C - (\mathscr{L}_{[\Theta^2 X, Y]} \Theta^2)^V\} - \\ - \{(\Theta^2)^C[X, \Theta^2 Y]^C - (\mathscr{L}_{[X, \Theta^2 Y]} \Theta^2)^V\} + \{(\Theta^4)^C[X, Y]^C - (\mathscr{L}_{[X, Y]} \Theta^4)^V\}.\end{aligned} \tag{3.10}$$

In consequence of (3.2), we have

$$(\Theta^2)^C (\Theta^2)^C = (\Theta^4)^C + (N_{\Theta^2, \Theta^2})^V. \tag{3.11}$$

Therefore by virtue of (3.11), equation (3.10) becomes

$$\begin{aligned}(N_{\Theta^2, \Theta^2}(X, Y))^C = [\Theta^2 X, \Theta^2 Y]^C - (\Theta^2)^C[\Theta^2 X, Y]^C - (\Theta^2)^C[X, \Theta^2 Y]^C + \\ + (\Theta^2)^C(\Theta^2)^C[X, Y]^C - (N_{\Theta^2, \Theta^2})^V[X, Y]^C + (\mathscr{L}_{[\Theta^2 X, Y]} \Theta^2 + \mathscr{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathscr{L}_{[X, Y]} \Theta^4)^V.\end{aligned} \tag{3.12}$$

Hence in view of (3.12), equation (3.8) yields

$$\begin{aligned}N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) = (N_{\Theta^2, \Theta^2}(X, Y))^C + (N_{\Theta^2, \Theta^2})^V[X, Y]^C - \\ - (\mathscr{L}_{[\Theta^2 X, Y]} \theta^2 + \mathscr{L}_{[X, \Theta^2 Y]} \Theta^2 - \mathscr{L}_{[X, Y]} \Theta^4)^V,\end{aligned}$$

which in consequence of (3.9) gives

$$N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) = (N_{\Theta^2, \Theta^2}(X, Y))^C + (N_{\Theta^2, \Theta^2})^V[X, Y]^C - \tilde{K}^V. \tag{3.13}$$

If $[X, Y]^C = 0$ and $\tilde{K}^V = 0$, then equation (3.13) reduces to

$$N_{(\Theta^4)^C, (\Theta^4)^C}(X^C, Y^C) = (N_{\Theta^2, \Theta^2}(X, Y))^C.$$

Theorem (3.3). *The Nijenhuis tensor of the complete lift of Θ^4 is equal to the complete lift of the Nijenhuis tensor of Θ^2 if*

(i) $$\mathscr{L}_X \Theta^2 = 0, \quad \mathscr{L}_Y \Theta^2 = 0$$

and

(ii) $$\mathscr{L}_X Y = 0, \quad \tilde{K}^V = 0.$$

Proof. Since $[X, Y]^C = 0$ implies that $[X, Y] = 0$ or $\mathscr{L}_X Y = 0$. Therefore from theorem (3.2), the result follows.

Theorem (3.4). *The processes of computing the Nijenhuis tensor of Θ^2 and taking the complete lift are commutative.*

PROOF. Since $\Theta^4-\Theta^2=0$. Hence the result is obvious.

4. *The horizontal lift of a* $\Theta(4, -2)$*-structure.*

Theorem (4.1). *Let* $\Theta \in \mathcal{J}_1^1(M)$ *be a* $\Theta(4, -2)$*-structure on* M. *Then the horizontal lift* Θ^H *of* Θ *is also a* $\Theta(4, -2)$*-structure on* $cT(M)$.

PROOF. For every Θ and Ψ such that $\Theta, \Psi \in \mathcal{J}_1^1(M)$, we have [3]

$$\Theta^H \psi^H + \psi^H \Theta^H = (\Theta\psi + \psi\Theta)^H. \tag{4.1}$$

Now replacing Ψ by Θ in (4.1), we obtain

$$(\Theta^H)^2 = (\Theta^2)^H. \tag{4.2}$$

Also replacing Ψ by Θ^2 in (4.1), we get

$$\Theta^H (\Theta^2)^H + (\Theta^2)^H \Theta^H = (2\Theta^3)^H,$$

which in view of (4.2) yields

$$(\Theta^H)^3 = (\Theta^3)^H. \tag{4.3}$$

Further replacing Ψ by Θ^3 in (4.1), we obtain

$$\Theta^H (\Theta^3)^H + (\Theta^3)^H \Theta^H = (2\Theta^4)^H,$$

which in consequence of (4.3) yields

$$(\Theta^H)^4 = (\Theta^4)^H. \tag{4.4}$$

Since Θ is a $\Theta(4, -2)$-structure on M. Therefore $\Theta^4-\Theta^2=0$. Hence from (4.2) and (4.4), we have

$$(\Theta^H)^4 = (\Theta^4)^H = (\Theta^2)^H = (\Theta^H)^2,$$

or

$$(\Theta^H)^4 - (\Theta^H)^2 = 0.$$

Thus Θ^H is a $\Theta(4, -2)$-structure on $cT(M)$.

Remark: The equation (4.4) can be generalised as follows:

$$(\Theta^H)^K = (\Theta^K)^H,$$

where K is some positive integer.

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