A new algebra of distributions on R^n

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Abstract. An algebra of distributions on R^n is introduced; it contains all the distributions having support in the positive cone $\{(t_1, \dots, t_n): t_i \geqq 0, i = 1, \dots, n\}$ and all the functions which are locally integrable on R^n. For $n=2$, the algebra consits of those distributions which are regular in the sets $\{(x, y): \eta < x < 0\}$ and $\{(x, y): \eta < y < 0\}$ for some $\eta < 0$. The multiplicative operation is convolution, defined so that there are no growth restrictions nor are there any restrictions on the supports. For those distributions having support in the positive cone, the given definition of convolution is consistent with the standard one.

In [1], [2] and [4] there is introduced a commutative algebra $\mathbf{B}$ of distributions on $(-\infty, \infty)$ which contains all the distributions having support in $[0, \infty)$ and all the functions which are locally integrable on $(-\infty, \infty)$. Each element of $\mathbf{B}$ has the property that it can be uniquely decomposed into left and right hand parts. The convolution $F \wedge G$ is defined so that there are no restrictions on the supports nor are there any growth restrictions; for those distributions having support in $[0, \infty)$ the definition of $F \wedge G$ is consistent with the usual operation of convolution [3, p. 112].

In this paper we extend the previous work to R^n. In this case, each element of $\mathbf{B}$ has the property that it can be uniquely decomposed into 2^n parts (for $n=2$, each element of $\mathbf{B}$ can be written as the sum of 4 distributions, each of whose supports is contained in one of the 4 quadrants). Convolution is defined so that, if f and g are continuous functions, then

$$(1) \qquad f \wedge g(t_1, \dots, t_n) = \int_0^{t_n} \dots \int_0^{t_1} f(t_1 - u_1, \dots, t_n - u_n) g(u_1, \dots, u_n)\, du_1 \dots du_n$$

for all $(t_1, \dots, t_n) \in R^n$, consistent with [4, (0.01)].

Throughout, n is a fixed positive integer and R^n denotes Euclidean n-space. For $k = 0, 1, \dots, 2^n - 1$, let

$$\sum_{i=1}^{n} \beta(k, i) 2^{i-1}$$

be the binary representation for k. Thus, each $\beta(k, i)$ is either 0 or 1. Define

$$X_k = \{(t_1, \dots, t_n) \in R^n : (-1)^{\beta(k, i)} t_i \geqq 0,\ i = 1, \dots, n\}$$

and

$$\sigma(k) = (-1)^{\sum_{i=1}^{n} \beta(k, i)}.$$

For example, if $n=1$ then X_0 is $[0, \infty)$ and X_1 is $(-\infty, 0]$; if $n=2$ then X_0, X_1, X_2 and X_3 are, respectively, the first, second, fourth and third quadrants. The numbers

$\sigma(0), \sigma(1), \sigma(2)$ and $\sigma(3)$ are, respectively, $+1, -1, -1$ and $+1$. For $i=1, \dots, n$ and any real number η we define

$$Y(\eta, i) = \{(t_1, \dots, t_n) \in R^n : t_i > \eta\}.$$

By a *distribution* (on R^n) we mean a continuous linear functional in the sense of LAURENT SCHWARTZ [3]. We define L to be the space of all complex-valued functions which are Lebesgue integrable on each bounded subset of R^n. If $f \in L$ then the regular distribution defined by f is denoted $[f]$. We say that the distribution F is *regular* in the open subset Ω of R^n if there exists $f \in L$ such that $F=[f]$ in Ω ([6, p. 25]). The *support* of a distribution F, denoted supp F, is the complement of the largest open set on which F vanishes.

Definition 1. We define $\mathbf{B}$ to be the set of distributions F on R^n for which there exist $\eta<0$ and distributions F_k $(k=0, 1, \dots, 2^n-1)$ such that

(*a*) $F = \sum_{k=0}^{2^n-1} F_k$;

(*b*) $\operatorname{supp} F_k \subset X_k \quad (k = 0, 1, \dots, 2^n-1)$;

(c) F_k is regular in the set $Y(\eta, i)$ if $1 \leqq k \leqq 2^n-1$ and $\beta(k, i)=1$.

Example. If F is a distribution whose support is contained in the "positive cone" X_0 then F belongs to $\mathbf{B}$.

Example. If $f \in L$ then $[f] \in \mathbf{B}$.

Theorem 1. *If F belongs to $\mathbf{B}$ then there exist unique distributions F_k $(k=0, 1, \dots, 2^n-1)$ which satisfy the conditions of Definition* 1.

We defer the proof of Theorem 1.

Notation. For each $F \in \mathbf{B}$ we denote by F^k $(k=0, 1, \dots, 2^n-1)$ the unique distributions F_k $(k=0, 1, \dots, 2^n-1)$ which satisfy the conditions of Definition 1.

Definition 2. If F and G belong to $\mathbf{B}$ we define

$$F \wedge G = \sum_{k=0}^{2^n-1} \sigma(k) F^k * G^k$$

where $F^k * G^k$ is the convolution of F^k and G^k [3, p. 112].

Remark. If F and G have their supports contained in the "positive cone" X_0 then $F^0=F$ and $G^0=G$; therefore, $F \wedge G$ is simply $F*G$.

Theorem 2. *If F and G belong to $\mathbf{B}$ then $F \wedge G$ belongs to $\mathbf{B}$ with $(F \wedge G)^k = F^k * G^k$ for each k.*

PROOF. Let $0 \leqq k \leqq 2^n-1$. If $0 \leqq (-1)^{\beta(k,i)} t_i < a$ and $0 \leqq (-1)^{\beta(k,i)} u_i < b$ for $i=1, \dots, n$, then $0 \leqq (-1)^{\beta(k,i)} (t_i+u_i) < a+b$ for $i=1, \dots, n$. By [3, Theorem 3, p. 113] the convolution $F^k * G^k$ is defined; from [3, Theorem 8, p. 120] we deduce that supp $F^k * G^k$ is contained in X_k. Suppose $\beta(k, i)=1$. There exist $\eta<0$ and functions f and g in L such that $F^k=[f]$ and $G^k=[g]$ in $Y(\eta, i)$. Letting $U=$ $=F^k-[f]$ and $V=G^k-[g]$ we have

$$F^k * G^k = U*V+U*[g]+[f]*V+[f]*[g].$$

Since $U=0=V$ in $Y(\eta, i)$ and $[f]=0=[g]$ in $Y(0, i)$ it follows from [3, Theorem 8, p. 120] that $U*V$, $U*[g]$ and $[f]*V$ are equal to zero in $Y(\eta, i)$ and therefore that $F^k*G^k=[f]*[g]=[f*g]$ in $Y(\eta, i)$; the second equality is from [3, Theorem 4, p. 114]. Thus, F^k*G^k is regular in $Y(\eta, i)$.

Theorem 3. *If F, G and H belong to* **B** *then $F\wedge G=G\wedge F$ and $(F\wedge G)\wedge H=$ $=F\wedge(G\wedge H)$.*

Proof. The equality $F\wedge G=G\wedge F$ comes from [3, Theorem 3, p. 113]. Let $0\leqq k\leqq 2^n-1$. If $0\leqq(-1)^{\beta(k,i)}t_i>a$, $0\leqq(-1)^{\beta(k,i)}u_i>b$ and $0\leqq(-1)^{\beta(k,i)}v_i<c$ for $i=1, \dots, n$, then

$$0 \leqq (-1)^{\beta(k,i)}(t_i+u_i+v_i) < a+b+c \quad (i=1, \dots, n).$$

From [3, p. 121] we deduce then that $(F^k*G^k)*H^k=F^k*(G^k*H^k)$. From Theorem 2 it follows that $(F\wedge G)^k*H^k=F^k*(G\wedge H)^k$ and therefore that $\big((F\wedge G)\wedge H\big)^k=$ $=\big(F\wedge(G\wedge H)\big)^k$.

Example. The Dirac distribution δ belongs to **B**. Note that $\delta^0=\delta$ and $\delta^k=0$ for $k>0$. Therefore, $\delta\wedge F=F^0$ for all $F\in\mathbf{B}$ (see [3, Theorem 6, p. 119]); if supp F is contained in the "positive cone" X_0 then $\delta\wedge F=F$.

Let $h\in L$ and $0\leqq k\leqq 2^n-1$. We define the function h^k to be equal to h on X_k and to be equal to 0 elsewhere; then $[h^k]=[h]^k$. If f and g belong to L we define

$$f\wedge g(t_1, \dots, t_n) = \sum_{k=0}^{2^n-1} \sigma(k) \int_{R^n}\!\!\dots\!\int f^k(t_1-u_1, \dots, t_n-u_n)g^k(u_1, \dots, u_n)\,du_1 \dots du_n.$$

By [3, Theorem 4, p. 114], the function $f\wedge g$ belongs to L and $[f\wedge g]=[f]\wedge[g]$. We observe that

$$f\wedge g(t_1, \dots, t_n) = \sigma(k) \int_{X_k}\!\!\dots\!\int f^k(t_1-u_1, \dots, t_n-u_n)g^k(u_1, \dots, u_n)\,du_1 \dots du_n$$

for $(t_1, \dots t_n)\in X_k$; therefore, if f and g are continuous, then (1) holds.

We conclude with the proof of our first theorem.

Proof *of Theorem 1.* Suppose $F\in\mathbf{B}$ and that F_k $(k=0, 1, \dots, 2^n-1)$ and G_k $(k=0, 1, \dots, 2^n-1)$ satisfy the conditions of Definition 1. We shall prove that $F_k=G_k$ for each k. By assumption, there exists $\eta<0$ such that F_k and G_k are regular in $Y(\eta, i)$ if $\beta(k, i)=1$. Suppose $\beta(k, i)=1$. There exist f and g in L such that $F_k=[f]$ and $G_k=[g]$ in $Y(\eta, i)$. Since $F_k=F=G_k$ in the interior of X_k, the functions f and g must be equal almost everywhere in the intersection of $Y(\eta, i)$ with the interior of X_k (see [5, Theorem 21.3]). Since $F_k=0=G_k$ in the complement of X_k, we have $f=0=g$ a.e. in the intersection of $Y(\eta, i)$ with the complement of X_k. Since the boundary of X_k has measure 0 we may conclude that $f=g$ a.e. in $Y(\eta, i)$. Again using [5, Theorem 21.3] we see that $[f]=[g]$ in $Y(\eta, i)$. We have shown then that $F_k=G_k$ in $Y(\eta, i)$ if $\beta(k, i)=1$. For each k define $N(k)$ to be the number of coefficients $\beta(k, i)$ $(i=1, \dots, n)$ for which $\beta(k, i)=0$. Thus, $N(2^n-1)=0$ while $N(0)=2^n$. We assert that $F_k=G_k$ for each k. This will be proved by induction on $N(k)$. Suppose $N(k)=0$. Then $\beta(k, i)=1$ for all i and therefore, by the above result, $F_k=G_k$ in $Y(\eta, i)$ for each i. But $F_k=F=G_k$ in the interior of X_k. Since the sets $Y(\eta, i)$ $(i=1, \dots, n)$ together with the interior of X_k form an open covering for R^n we may use [5, Theorem 24.1] to conclude that $F_k=G_k$. Thus, the assertion is true for $N(k)=0$. Assume $F_k=G_k$ for all k satisfying $N(k)\leqq p$. Suppose $N(k)=$ $=p+1$, where $p+1\leqq 2^n$. Let $k_1, \dots, k_s$ be those indices for which $\beta(k, i)\leqq\beta(k_r, i)$ for all i. Then $N(k_r)\leqq p$ for each r and

$$F_k+F_{k_1}+\dots+F_{k_s} = F = G_k+G_{k_1}+\dots+G_{k_s}$$

in the set

$$Y = \{(t_1, \dots, t_n) : t_i < 0 \text{ if } \beta(k, i) = 1\}.$$

But, by our induction hypothesis, $F_{k_r}=G_{k_r}$ for $r=1, \dots, s$. Therefore, $F_k=G_k$ in the set Y. But $F_k=G_k$ in each of the sets $Y(\eta, i)$ where $\beta(k, i)=1$. Since the set Y together with each of the sets $Y(\eta, i)$, where $\beta(k, i)=1$, form an open covering for R^n, we have $F_k=G_k$. The assertion is thus true for all k satisfying $N(k)=$ $=p+1$ and the induction proof is completed.

Bibliography

[1] G. Krabbe, An algebra of generalized functions on an open interval; two-sided operational calculus, *Bull Amer. Math. Soc.* **77** (1971), 78—84.

[2] G. Krabbe, An algebra of generalized functions on an open interval: two-sided operational calculus, *Pacific J. Math.* **42** (1972), 381—411.

[3] L. Schwartz, Mathematics for the physical sciences, *Reading*, 1966.

[4] H. Shultz, An algebra of distributions on an open interval, *Trans. Amer. Math. Soc.* **169** (1972), 163—181.

[5] F. Treves, Topological vector spaces, distributions and kernels, *New York*, 1967.

[6] A. Zemanian, Distribution theory and transform analysis, *New York*, 1965.

(Received July 9, 1975.)