The radius of starlikeness of certain subclasses of analytic functions

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1. *Introduction:* The sharp lower bounds for $\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\}$, radius of starlikeness etc. have been determined for the class $Q(\alpha, \lambda)$ of analytic functions $F(z)$ in $E(|z|<1)$; $F(z)=(1-\lambda)z+\lambda f(z)$, $0\leqq\lambda\leqq 1$, where $f(z)$ belongs to the class $R(\alpha)$ of analytic functions

$$f(z) = z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

satisfying the condition $\operatorname{Re}\left\{\frac{f(z)}{z}\right\}>\alpha$, $0\leqq\alpha<1$ there, thus, leading to some good observations about the classes $R(\alpha)$, $Q(\alpha, \lambda)$ etc.

Let S denote the class of analytic *schlicht (univalent)* functions represented by (1.1). For fixed α, $0\leqq\alpha<1$, let $S^*(\alpha)$ denote the class of normalized functions $f(z)$, given by (1.1) which are analytic, schlicht and starlike of order α in E, so that the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad \text{holds for} \quad z\in E. \tag{1.2}$$

For fixed $\alpha, 0\leqq\alpha<1$, let $C(\alpha)$ denote the class of normalized functions $f(z)$ of the form (1.1), which are analytic schlicht and convex of order α in E i.e. the functions $f(z)$ satisfying the condition

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] > \alpha \quad \text{for} \quad z\in E. \tag{1.3}$$

Let $P(\alpha)$ be the class of functions

$$p(z) = 1+\sum_{n=1}^{\infty} a_n z^n \tag{1.4}$$

which are analytic in E and have real part greater than α, $0\leqq\alpha<1$ in E i.e. $p(z)$ satisfying the properties:

$$\operatorname{Re} p(z) > \alpha \quad \text{in } E \text{ and} \quad p(0) = 1.$$

Let $R(\alpha)$ be the class of analytic functions $f(z)$ of the form (1.1), satisfying the property

(1.5) $$\operatorname{Re}\left[\frac{f(z)}{z}\right] > \alpha \quad \text{in } E \text{ i.e. } \frac{f(z)}{z} \in P(\alpha).$$

Next, denote by $Q(\alpha, \lambda)$, the class of functions

(1.6) $$F(z) = \lambda f(z) + (1-\lambda)z$$

where $f(z) \in R(\alpha)$, $0 \leqq \lambda \leqq 1$, $0 \leqq \alpha < 1$.
Such classes have been considered from the point of view to determine the radii of schlichtness, starlikeness and convexity of a linear combination of mappings, having certain properties of starlikeness. Convexity or real part greater than $\alpha (0 \leqq \alpha < 1)$ etc.

The class $R(\alpha)$ has been considered as regards finding out the sharp lower bound for $\operatorname{Re} f'(z)$, first by YAMAGUCHI (17] for $\alpha = 0$ and later by GOEL [3], with second coefficient fixed, giving an improvement of the former result. The univalency of the partial sums were also studied there.

Our class $Q(\alpha, \lambda)$ discusses Hayman's question [6] for the special case, where the 2nd function becomes an identity function. This case may be viewed from the following stand point:

Suppose that $f(z) \in C(0)$ and that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $F(z) = \lambda f(z) + (1-\lambda)z = z + \lambda \sum_{n=2}^{\infty} a_n z^n$. *What can be said about the schlichtness or starlikeness of* $F(z)$? In a recent paper, TRIMBLE [16] proved that if $f(z) \in C(0)$, then $F(z)$, represented by (1.6), has the following properties:

(i) $F(z) \in K$; K being the class of close to convex functions, introduced by KAPLAN [8],

(ii) $F(z) \in S^*\left(\frac{3\lambda - 2}{2(2-\lambda)}\right)$ if $\frac{2}{3} \leqq \lambda < 1$,

(iii) $F(z)$ need not be in $S^*(0)$ for $\lambda < 2/3$.

Afterwards, CHICHERA and SINGH [2] generalized the case by attaining that $F(z)$ becomes starlike for all λ, $0 \leqq \lambda \leqq 1$, if certain additional restriction be imposed on $f(z)$ i.e. "if $f(z) \in C(0)$, then $F(z) = \lambda \frac{2}{z} \int_0^z f(t)\,dt + (1-\lambda)z$ belongs to $S^*(0)$ for all λ, $0 \leqq \lambda \leqq 1$". In our case, the class $Q(\alpha, 1)$ coincides with $R(\alpha)$. We mention an interesting result, derived by GOPAL [4], on using variational techniques of ROBERTSON [12].

"Let $f(z) \in S$ and $f(z) \in R(\alpha)$, $0.1 \leqq \alpha \leqq 1$, then $f(z)$ is starlike for $|z| < r_\alpha$, where $r_\alpha = \frac{\sqrt{(1-\alpha)} - \alpha}{1 - 2\alpha}$. This result is sharp (this generalizes a result due to MACGREGOR [9])".

In this context, we deduce the above result under the weaker assumption when $f(z)$ *need not be schlicht in* E *and also go through some interesting observations for the classes* $R(\alpha)$, $Q(\alpha, \lambda)$ *etc ...*

2. *Useful lemmas*

First of all, we shall like to mention the following lemmas, equally helpful in this direction, derivable on the basis of the *basic lemma of* SINGH *and* BAJPAI [1].
Lemma A [1] Let

$$H(z)=\frac{a}{1+z\varphi(z)}-\frac{1}{(1+bz\varphi(z))}-\frac{(1-b)z^2\varphi^1(z)}{(1+z\varphi(z))(1+bz\varphi(z))} \tag{2.1}$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\leqq 1$ in E, $-1<b\leqq 1$ and $a\geqq 1$ then for $|z|=r$, $0\leqq r<1$,

$$\operatorname{Re} H(z)\leqq\frac{(a-1)+(1-ab)r}{(1-r)(1-br)}, \tag{2.2}$$

$$\operatorname{Re} H(z)\geqq \tag{2.3}$$

$$\geqq\begin{cases}-\dfrac{(1+ab+2b)(1-r^2)+2(1-b)}{(1-b)(1-r^2)}+\dfrac{2}{1-b}\sqrt{\dfrac{(1+a)(1+b)(1-br^2)}{1-r^2}} & \text{for}\quad u_0\geqq u_1\\[2ex] \dfrac{(a-1)+(ab-1)r}{(1+r)(1+br)} & \text{for}\quad u_0\leqq u_1\end{cases}$$

where $u_0=\dfrac{1}{1-b}\left[\sqrt{\dfrac{(1+b)(1-br^2)}{(1+a)(1-r^2)}}-b\right]$ and $u_1=\dfrac{1}{1+r}$.

Lemma 1. *Let λ satisfy $0\leqq\lambda\leqq 1$ and $f(z)\in Q(\alpha,\lambda)$ i.e.*

$$F(z)=(1-\lambda)z+\lambda f(z),\quad \textit{where}\quad f(z)\in R(\alpha),\quad 0\leqq\alpha<1.$$

Let $r_1(\alpha,\lambda)$ denote the smallest positive root of the equation

$$-1+3r+3(1+2\alpha\lambda-2\lambda)r^2-(1+2\alpha\lambda-2\lambda)r^3=0 \tag{2.4}$$

which exists in $(0, 1)$, *then for* $|z|=r$, $0\leqq r<1$, *we have*

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right]\leqq\frac{1-2(1+2\alpha\lambda-2\lambda)r+(1+2\alpha\lambda-2\lambda)r^2}{(1-r)(1-(1+2\alpha\lambda-2\lambda)r)} \tag{2.5}$$

(2.6)

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right]\geqq\begin{cases}\dfrac{1+2(1+2\alpha\lambda-2\lambda)r+(1+2\alpha\lambda-2\lambda)r^2}{(1+r)(1+(1+2\alpha\lambda-2\lambda)r)} & \text{for}\quad 0\leqq r\leqq r_1(\alpha,\lambda)\\[2ex] \dfrac{1}{\lambda(1-\alpha)}\left[\sqrt{4(1+\alpha\lambda-\lambda)A}-(1+2\alpha\lambda-2\lambda)-A\right] & \text{for}\quad r_1(\alpha,\lambda)\leqq r<1\end{cases}$$

where $A=\dfrac{1-(1+2\alpha\lambda-2\lambda)r^2}{1-r^2}$. *Equality sign in* (2.5) *and* the *first inequality of* (2.6) *is attained for the function*

$$f(z)=z\,\frac{1+(2\alpha-1)z}{1+z},\quad \textit{for}\quad 0\leqq r\leqq r_1(\alpha,\lambda) \tag{2.7}$$

and, that in the second inequality corresponding to the function $f(z)$, satisfying the conditions:

$$f(z) = z\frac{1-2b\alpha z+(2\alpha-1)z^2}{1-2bz+z^2}, \tag{2.8}$$

where b is determined from the relation:

$$\frac{1-(2+2\alpha\lambda-2\lambda)br+(1+2\alpha\lambda-2\lambda)r^2}{1-2br+r^2} = \sqrt{(1+\alpha\lambda-\lambda)A} \equiv R_0. \tag{2.9}$$

PROOF. Since $f(z)\in R(\alpha)$, there exists a function $\varphi(z)$, satisfying schwarz's lemma, such that

$$\frac{f(z)}{z} = \frac{1+(2\alpha-1)z\varphi(z)}{1+z\varphi(z)}. \tag{2.10}$$

Consequently, we obtain

$$F(z) = z\left[\frac{1+(1+2\alpha\lambda-2\lambda)z\varphi(z)}{1+z\varphi(z)}\right]. \tag{2.11}$$

Diff. logarithmically $F(z)$ w.r.t. z and then using (2.10) and simplyfying,

(2.12)

$$\frac{zF'(z)}{F(z)} = 1+\frac{1}{1+z\varphi(z)}-\frac{1}{1+(1+2\alpha\lambda-2\lambda)z\varphi(z)}-\frac{2\lambda(1-\alpha)z^2\varphi^1(z)}{(1+z\varphi(z))(1+(1+2\alpha\lambda-2\lambda)z\varphi(z))}.$$

Using *lemma A*, with $a=1$ and $b=1+2\alpha\lambda-2\lambda$, we deduce (2.5) and moreover,

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right] \geqq \begin{cases} \dfrac{1}{\lambda(1-\alpha)}\left[\sqrt{4(1+\alpha\lambda-\lambda)A}-(1+2\alpha\lambda-2\lambda)-A\right] & \text{for } \underline{u_0 \geqq u_1} \\ \dfrac{1+2(1+2\alpha\lambda-2\lambda)r+(1+2\alpha\lambda-2\lambda)r^2}{(1+r)(1+(1+2\alpha\lambda-2\lambda)r)} & \text{for } \underline{u_0 \leqq u_1} \end{cases} \tag{2.13}$$

where $u_0 = \dfrac{1}{2\lambda(1-\alpha)}\left[\sqrt{(1+\alpha\lambda-\lambda)A}-(1+2\alpha\lambda-2\lambda)\right]$ and $u_1 = \dfrac{1}{1+r}$.

The two inequalities in (2.13) become equal for such values of λ and α, for which $u_0=u_1$,

$$\text{i.e.}\quad \frac{(1+\alpha\lambda-\lambda)(1-(1+2\alpha\lambda-2\lambda)r^2)}{1-r} = \frac{(1+(1+2\alpha\lambda-2\lambda)r)^2}{1+r},$$

i.e. $$g(\alpha, \lambda, r) \equiv -1+3r+3(1+2\alpha\lambda-2\lambda)r^2-(1+2\alpha\lambda-2\lambda)r^3 = 0.$$

Then, it is easy to check that $g(\alpha, \lambda, r)$ is a strictly increasing function of r, $0\leqq r<1$, for each $\alpha, \lambda; 0\leqq\lambda\leqq 1, 0\leqq\alpha<1$.

Also $$g(\alpha, \lambda, 0) = -1 < 0 \quad\text{and}\quad f(\alpha, \lambda, 1) = 4[1-\lambda(1-\alpha)] > 0.$$

Hence $g(\alpha, \lambda, r)$ changes sign from negative to positive while moving through (0, 1), so that the equation $g(\alpha, \lambda, r)=0$ has a unique root $r_1(\alpha, \lambda)$ in (0, 1). The proof is now complete. The sharpness of the result follows on using the technique of SINGH and BAJPAI [1] or SINGH and GOEL [13].

Corollary 1. Let λ satisfy $0 \leqq \lambda \leqq 1$ and $F(z) \in Q\left(\frac{1}{2}, \lambda\right)$ i.e.

$$F(z) = (1-\lambda)z + \lambda f(z), \quad \text{where} \quad f(z) \in R\left(\frac{1}{2}\right).$$

Let $r_1(\lambda)$ denote the smallest positive root of the equation, which is unique in $(2-\sqrt{3}, 1]$,

$$-1+3r+3(1-\lambda)r^2-(1-\lambda)r^3 = 0. \tag{2.14}$$

Then, for $|z|=r$, $0 \leqq r < 1$, we have

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right] \leqq \frac{1-2(1-\lambda)r+(1-\lambda)r^2}{(1-r)(1-(1-\lambda)r)} \tag{2.15}$$

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right] \geqq \begin{cases} \dfrac{2}{\lambda}\left[\sqrt{2(2-\lambda)A}-(1-\lambda)-A\right] & \text{for} \quad \underline{r_1(\lambda) \leqq r < 1} \\[2ex] \dfrac{1+2(1-\lambda)r+(1-\lambda)r^2}{(1+r)(1+(1-\lambda)r)} & \text{for} \quad \underline{0 \leqq r \leqq r_1(\lambda)}. \end{cases} \tag{2.16}$$

Equality sign in (2.15) and the second inequality of (2.16) is attained for the function:

$$F(z) = \frac{z}{1+z}, \quad 0 \leqq r \leqq r_1(\lambda) \tag{2.17}$$

and, that, in the first inequality of (2.16) corresponding to the function $f(z)$, satisfying

$$f(z) = z\,\frac{1-bz}{1-2bz+z^2} \tag{2.18}$$

where b is determined from the relation

$$\frac{1-(2-\lambda)br+(1-\lambda)r^2}{1-2br+r^2} = \sqrt{\frac{2-\lambda}{2}}\,A \equiv R_0.$$

Remark 1. If $F(z)=(1-\lambda)z+\lambda f(z)$, where $f(z)\in C(0)$, then, still the corollary 1 holds, since $f(z)\in C(0)$ implies that $f(z)\in R(1/2)$.

For $\lambda=1$, we derive the upper and lower bounds of $\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]$, when $f(z)\in R(\alpha)$, $0\leqq\alpha<1$, as compared to SHAFFER [14, 15].

Corollary 2. Let $f(z)\in R(\alpha)$, $0\leqq\alpha<1$ and $r_1(\alpha)$ denotes the smallest positive root of the equation, which is unique in $(2-\sqrt{3}, 1]$:

$$-1+3r+3(2\alpha-1)r^2-(2\alpha-1)r^3 = 0. \tag{2.19}$$

Then, for $|z|=r,\ 0\leqq r<1$, we have

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] \leqq \frac{1-2(2\alpha-1)r+(2\alpha-1)r^2}{(1-r)(1-(2\alpha-1)r)} \tag{2.20}$$

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] \geqq \begin{cases} \dfrac{1+2(2\alpha-1)r+(2\alpha-1)r^2}{(1+r)(1+(2\alpha-1)r)} & \text{for} \quad \underline{0\leqq r\leqq r_1(\alpha)} \\ \dfrac{1}{1-\alpha}\left[\sqrt{4\alpha A}-(2\alpha-1)-A\right] & \text{for} \quad \underline{r_1(\alpha)\leqq r<1} \end{cases} \tag{2.21}$$

where $A=\dfrac{1-(2\alpha-1)r^2}{1-r^2}$. Equality sign in (2.20) and the first inequality of (2.21) occur for the function (2.7) and that in the second inequality for the function (2.8), where b is determined from the relation:

$$\frac{1-2\alpha br+(2\alpha-1)r^2}{1-2br^2+r^2} = \sqrt{\alpha A} \equiv R_0. \tag{2.22}$$

Remark 2. The class $R(\alpha)$ for suitable α, can be generated either with the help of the classes $C(\alpha)$ or $S^*(\alpha)$, $0\leqq\alpha<1$ as mentioned by MARX [11], JACK [7], MACGREGOR & coauthors [10], HALLENBECK [5] etc.

3. *Radius of Starlikeness for the Class $Q(\alpha, \lambda)$*

Theorem 1. *Let $F(z)\in Q(\alpha,\lambda)$ and $r_1(\alpha,\lambda)$ be the smallest positive root, which exists in* (0, 1), *of the equation* (2.4), *then $F(z)$ is starlike for $|z|<r_0(\alpha,\lambda)$, where*

$$r_0(\alpha,\lambda) = \left[\frac{1+\alpha\lambda-\lambda}{(1+\alpha\lambda-\lambda)+\sqrt{\lambda(1-\alpha)(1+\alpha\lambda-\lambda)}}\right]^{1/2} \tag{3.1}$$

and moreover, $r_0(\alpha,\lambda)\geqq r_1(\alpha,\lambda)$. This result is sharp.

For the class $Q\left(\frac{1}{2},\lambda\right)$, the above theorem takes the form:

Corollary 1. Let $F(z)\in Q\left(\frac{1}{2},\lambda\right)$ and $r_1(\lambda)$ be the smallest positive root of the equation (2.14), which exists in $(2-\sqrt{3}, 1]$, then $F(z)$ is starlike for $|z|<r_0(\lambda)$, where

$$r_0(\lambda) = \left[\frac{2-\lambda}{(2-\lambda)+\sqrt{\lambda(2-\lambda)}}\right]^{1/2} \quad \text{and} \quad r_0(\lambda)\geqq r_1(\lambda). \tag{3.2}$$

This result is sharp.

For the class $Q(\alpha, 1)\equiv R(\alpha)$, we deduce the following:

Corollary 2. Let $f(z)\in R(\alpha)$ and $r_1(\alpha)$ denotes the smallest positive root of the equation (2.19), which is unique in $(2-\sqrt{3}, 1]$, then $f(z)$ is starlike for $|z|<r_0(\alpha)$, where

$$r_0(\alpha) = \left[\frac{\alpha}{\alpha+\sqrt{(1-\alpha)}}\right]^{1/2} \tag{3.3}$$

$(0<\alpha<1)$. This result is sharp.

The above corollary is also obtained by GOPAL [4] on using different techniques, with the additional condition that $f(z)\in S$. Hence, our result shows an improvement.
The above theorem follows immediately as a consequence of the *lemma* 1.

Corollary 3 [9]. Let $f(z)\in R\left(\frac{1}{2}\right)$, then $f(z)$ is schlicht and starlike for $|z|<\frac{1}{\sqrt{2}}$. This result is sharp.

Remark 3. After some tedius manpulations, it can be seen that the function $F(z)\in Q(\alpha,\lambda)$ in *theorem* 1, is starlike of order β, $0\leqq\beta<1$ for $|z|<r_0(\alpha,\lambda,\beta)$, where

$$r_0(\alpha,\lambda,\beta)=\left[\frac{\sqrt{4\lambda(1-\alpha)(1-\beta)(1+\alpha\lambda-\lambda)}-\beta\lambda(1-\alpha)}{\lambda(1-\alpha)(2-\beta)+\sqrt{4\lambda(1-\alpha)(1-\beta)(1+\alpha\lambda-\lambda)}}\right]^{1/2}. \tag{3.4}$$

This result is also sharp.

Observations. Some sort of invariance property can be observed for the classes $R(\alpha)$, $Q(\alpha,\lambda)$ etc.

(i) If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in R(\alpha)$, $0\leqq\alpha<1$, then for each λ, $0\leqq\lambda\leqq 1$, $h_\lambda(z)=z+\lambda\sum_{n=2}^{\infty}a_nz^n\in R(\alpha)$.

Since $h_\lambda(z)=\lambda f(z)+(1-\lambda)z$, we have

$$\operatorname{Re}\left[\frac{h_\lambda(z)}{z}\right]=\lambda\operatorname{Re}\left[\frac{f(z)}{z}\right]+(1-\lambda)>\lambda\alpha+(1-\lambda)\geqq\alpha.$$

Hence, $h_\lambda(z)\in R(\alpha)$, as required.
Explicitly, if $F(z)\in Q(\alpha,\lambda)$, then $F(z)\equiv h_\lambda(z)=\lambda f(z)+(1-\lambda)z$ and therefore, $h_\lambda(z)\in R(\alpha)$.
Now, consider the class $Q_1(\alpha,\lambda)$, consisting of the functions $h_\lambda^{(1)}(z)$ as defined by: $h_\lambda^{(1)}(z)=(1-\lambda)z+\lambda\cdot h_\lambda(z)$, where $h_\lambda(z)\in\beta(\alpha,\lambda)$.
Therefore

$$\frac{h_\lambda^{(1)}(z)}{z}=(1-\lambda)+\lambda\frac{h_\lambda(z)}{z}=(1-\lambda^2)+\lambda^2\frac{f(z)}{z}.$$

So that

$$\operatorname{Re}\left[\frac{h_\lambda^{(1)}(z)}{z}\right]=\lambda^2\operatorname{Re}\left[\frac{f(z)}{z}\right]+(1-\lambda^2)\geqq\alpha$$

and hence $h_\lambda^{(1)}(z)\in R(\alpha)$.
In this manner, we can construct, from the given class $R(\alpha)$, respective subclasses $Q(\alpha,\lambda)$, $Q_1(\alpha,\lambda)$, $Q_2(\alpha,\lambda)$, ..., so that the sequence $\langle Q_n(\alpha,\lambda)\rangle$ of classes satisfies the property: $Q_n(\alpha,\lambda)\subseteqq Q_{n+1}(\alpha,\lambda)$ for $n\geqq 1$.

(ii) For the class $Q(\alpha,\lambda)$, some sort of converse problem is solved i.e. if

$$F(z)=(1-\lambda)z+\lambda f(z),$$

where $F(z) \in R(\alpha)$, then

(3.5) $f(z) = \frac{1}{\lambda}[F(z)-(1-\lambda)z], \quad 0 < \lambda \leqq 1$ (for $\lambda = 0$, we see that $F(z) \equiv z$).

Since $F(z) \in R(\alpha)$, we have

$$\frac{F(z)}{z} = \frac{1+(2\alpha-1)z\varphi(z)}{1+z\varphi(z)}, \quad \text{where} \quad \varphi(z) \tag{3.6}$$

satisfies the conditions of Schwarz's lemma.

Using (3.6) in (3.5), we obtain

$$f(z) = z\frac{1+bz\varphi(z)}{1+z\varphi(z)}, \quad \text{where} \quad b = 1-\frac{2(1-\lambda)}{\lambda}, \tag{3.7}$$

Diff. logarithmically $f(z)$ w.r.t. z provides

$$\frac{zf'(z)}{f(z)} = 1+\frac{1}{1+z\varphi(z)}-\frac{1}{1+bz\varphi(z)}-\frac{(1-b)z^2\varphi^1(z)}{(1+z\varphi(z))(1+bz\varphi(z))}. \tag{3.8}$$

Now, for $a=1$ and $b=1-\frac{2(1-\alpha)}{\lambda}$ in the range $-1<b\leqq 1$, we can apply lemma A, to determine the lower and upper bounds of $\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]$, from which the radius of starlikeness can be obtained in an analogous manner.

4. *Radius of starlikeness for the class* $N(\alpha, \lambda)$

After seeing various properties of the class $Q(\alpha, \lambda)$, it is natural to construct a new class $N(\alpha, \lambda)$, which consists of the functions of the type.

$$F(z) = \frac{z}{g(z)}[(1-\lambda)z+\lambda f(z)] \tag{4.1}$$

where $g(z) \in C(0)$, $S^*(\beta)$ or $R(\beta)$ and $f(z) \in R(\alpha)$ $(0 \leqq \alpha, \beta < 1;\ 0 \leqq \lambda \leqq 1)$. We use the notations given below

$$\Theta_1(r, \alpha, \lambda) = \frac{1+2(1+2\alpha\lambda-2\lambda)r+(1+2\alpha\lambda-2\lambda)r^2}{(1+r)(1+(1+2\alpha-2\lambda)r)},$$

$$\Theta_2(r, \alpha, \lambda) = \frac{1}{\lambda(1-\alpha)}\left[\sqrt{4(1+\alpha\lambda-\lambda)A}-(1+2\alpha\lambda-2\lambda)-A\right],$$

where

$$A = 1-\frac{(1+2\alpha\lambda-2\lambda)r^2}{1-r^2},$$

$$\mu(r, \alpha) = \frac{1-(2\alpha-1)r}{1-r},$$

$$\eta(r, \alpha) = \frac{1-2(2\alpha-1)r+(2\alpha-1)r^2}{(1-r)(1-(2\alpha-1)r)}.$$

Theorem 2. *Let $F(z)$ be represented by* (4.1), *where $f(z)\in R(\alpha)$, $0\leqq\alpha<1$ and $g(z)\in C(0)$, then $F(z)$ is starlike for $|z|<r_0<1$, r_0 is the smallest positive root of the equation*

$$\overline{T}(r, \alpha, \lambda) = 0, \tag{4.2}$$

where

$$\overline{T}(r, \alpha, \lambda) = \begin{cases} 1-\dfrac{1}{1-r}+\Theta_1(r, \alpha, \lambda); & 0 \leqq r \leqq r_1(\alpha, \lambda), \\ 1-\dfrac{1}{1-r}+\Theta_2(r, \alpha, \lambda); & r_1(\alpha, \lambda) \leqq r < 1, \end{cases}$$

$r_1(\alpha, \lambda)$ denotes the smallest positive root of the equation (2.4).

Proof. Since $f(z)\in R(\alpha)$, we have $\dfrac{f(z)}{z}\in P(\alpha)$, and hence for some $\Phi(z)$ we can write

$$F(z) = \frac{z^2}{g(z)}\left[\frac{1+(1+2\alpha\lambda-2\lambda)z\Phi(z)}{1+z\Phi(x)}\right], \tag{4.3}$$

where $\Phi(z)$ satisfies Schwarz's lemma i.e. $\Phi(0)=0$, and $|\Phi(z)|\leqq 1$ for $z\in E$. Differentiating $F(z)$ logarithmically with respect to z we get

$$\frac{zF'(z)}{F(z)} = 1-\frac{zg'(z)}{g(z)}+1-\frac{2\lambda(1-\alpha)\big(z\Phi(z)+z^2\Phi'(z)\big)}{(1+z\Phi(z))(1+(1+2\alpha\lambda-\lambda)z\Phi(z))}. \tag{4.4}$$

Now, on using *Lemma* 1 and the fact that $\left|\dfrac{zg'(z)}{g(z)}\right|\leqq\dfrac{1}{1-r}$ for $g(z)\in C(0)$, it follows that

$$\operatorname{Re}\left[\frac{zF'(z)}{F(z)}\right] \geqq \begin{cases} 1-\dfrac{1}{1-r}+\Theta_1(r, \alpha, \lambda), & 0 \leqq r \leqq r(\alpha, \lambda), \\ 1-\dfrac{1}{1-r}+\Theta_2(r, \alpha, \lambda) & r_1(\alpha, \lambda) \leqq r < 1. \end{cases}$$

Since the bounds used here are sharp, so the results of this theorem are sharp.
The proofs of the following theorems are similar to that of the above and hence, can be omitted.

Theorem 3. *Let $F(z)$ be represented by* (4.1) *where $f(z)\in R(\alpha)$, $0\leqq\alpha<1$ and $g(z)\in S^*(\beta)$, $0\leqq\beta<1$, then $F(z)$ is starlike for $|z|<r_0<1$, where r_0 is the smallest positive root of the equation*

$$\overline{T}(r, \alpha, \beta, \lambda) = 0 \tag{4.6}$$

where

$$\overline{T}(r, \alpha, \beta, \lambda) = \begin{cases} 1-\mu(r, \beta)+\Theta_1(r, \alpha, \lambda), & 0 \leqq r \leqq r_1(\alpha, \lambda), \\ 1-\mu(r, \beta)+\Theta_2(r, \alpha, \lambda), & r_1(\alpha, \lambda) \leqq 1 < r. \end{cases}$$

$r_1(\alpha, \lambda)$ is the same as stated in Theorem 2. This result is sharp.

Theorem 4. *Let $F(z)$ be represented by (4.1), where $f(z)\in R(\alpha)$, $0\leqq\alpha<1$ and $g(z)\in R(\beta)$, $0\leqq\beta<1$, then $F(z)$ is starlike for $|z|<r_0<1$, where r_0 is the smallest positive root of the equation:*

$$\overline{T}(r, \alpha, \beta, \lambda) = 0,$$

$$\overline{T}(r, \alpha, \beta, \lambda) = \begin{cases} 1+\overline{\Theta}_1(r, \alpha, \lambda)-\eta(r, \beta), & 0 \leqq r \leqq r_1(\alpha, \beta) \\ 1+\overline{\Theta}_2(r, \alpha, \lambda)-\eta(r, \beta), & r_1(\alpha, \lambda) \leqq r < 1 \end{cases}$$

where $r_1(\alpha, \lambda)$ is the same as in Theorem 2. This result is sharp.

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