Projective symmetry in a special Kawaguchi space

By H. S. SHUKLA (Gorakhpur) and H. D. PANDE (Jabalpur).

1. Summary and introduction

M. C. CHAKI and B. GUPTA [1] have defined an n-dimensional conformally symmetric Riemannian space. In one of her papers, B. GUPTA [2] has studied various properties of the projectively symmetric Riemannian spaces. The object of this paper is to define an n-dimensional projectively symmetric special Kawaguchi space and discuss its properties.

An n-dimensional metric space, in which the arc length of a curve $x^i = x^i(t)$ is given by the special integral

$$s = \int F^{1/p}\,dt, \quad F \equiv A_i(x, x')x''^i + B(x, x'), \tag{1.1}$$

where $A_i(x, x')$ and $B(x, x')$ are differentiable functions of x, x', is called an n-dimensional special Kawaguchi space. A. KAWAGUCHI ([3], [4]) has defined a connection in a special Kawaguchi space by applying the "Craig vector" of the function $A_i x''^i + B$. The Craig vector is defined as

$$T_i = (A_{k(i)} - 2A_{i(k)})x''^k - 2A_{ik}x'^k + B_{(i)}, \tag{1.2}$$

where
$$A_{k(i)} = \frac{\partial A_k}{\partial x'^i}, \quad A_{ik} = \frac{\partial A_i}{\partial x^k}, \quad B_{(i)} = \frac{\partial B}{\partial x'^i}.$$

There exists a relation

$$\begin{cases} F = A_i x^{[2]i}, \quad \text{where} \quad 2p \neq 3, \\ x^{[2]i} = x''^i + 2\Gamma^i, \quad 2\Gamma^i = (2A_{lk}x'^k - B_{(l)})G^{li}, \\ G_{ik} = 2A_{i(k)} - A_{k(i)}, \quad G_{ik}G^{il} = \delta^l_k. \end{cases} \tag{1.3}$$

The covariant derivative of a contravariant vector field v^i, which is homogeneous of degree zero with respect to x'^i, is defined by

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} - \frac{\partial v^i}{\partial x'^k}\Gamma^k_{(j)} + \Gamma^i_{(k)(j)}v^k, \quad \nabla'_j v^i = \frac{\partial v^i}{\partial x'^j}. \tag{1.4}$$

Then from the brackets of Poisson for covariant derivatives, we find the curvature tensor fields as follows:

$$(\nabla_j\nabla_k - \nabla_k\nabla_j)v^i = -R_{jkl}{}^{\cdot\cdot\cdot i}v^l + K_{jk}{}^{\cdot\cdot l}\nabla'_l v^i, \tag{1.5}$$

$$(\nabla_j\nabla'_k - \nabla'_k\nabla_j)v^i = -B_{jkl}{}^{\cdot\cdot\cdot i}v^l, \tag{1.6}$$

where

(1.7) $$B_{jkl}{}^{i} = \Gamma^{i}_{(j)(k)(l)},$$

(1.8)
$$R_{jkl}{}^{i} = \frac{\partial \Gamma^{i}_{(l)(j)}}{\partial x^{k}} - \frac{\partial \Gamma^{i}_{(l)(k)}}{\partial x^{j}} + \Gamma^{h}_{(l)(j)}\Gamma^{i}_{(k)(h)} - \Gamma^{h}_{(l)(k)}\Gamma^{i}_{(j)(h)} + \Gamma^{h}_{(j)}\Gamma^{i}_{(l)(k)(h)} - \Gamma^{h}_{(k)}\Gamma^{i}_{(l)(j)(h)},$$

(1.9) $$K_{jk}{}^{i} = \frac{\partial \Gamma^{i}_{(j)}}{\partial x^{k}} - \frac{\partial \Gamma^{i}_{(k)}}{\partial x^{j}} + \Gamma^{h}_{(j)}\Gamma^{i}_{(k)(h)} - \Gamma^{h}_{(k)}\Gamma^{i}_{(j)(h)}.$$

These curvature tensor fields satisfy the following identities:

(1.10) $$R_{jkl}{}^{i} = -R_{kjl}{}^{i}, \quad R_{jkl}{}^{i} + R_{klj}{}^{i} + R_{ljk}{}^{i} = 0,$$

(1.11) $$K_{jk}{}^{i} = R_{jkl}{}^{i}x'^{l}, \quad R_{jkl}{}^{i} = K_{jk}{}^{i}{}_{(l)} = \nabla'_{l}K_{jk}{}^{i},$$

(1.12) $$B_{jkl}{}^{i}x'^{l} = 0.$$

The Bianchi identity for $K_{jk}{}^{i}$ is given by

(1.13) $$\nabla_{h}K_{jk}{}^{i} + \nabla_{j}K_{kh}{}^{i} + \nabla_{k}K_{hj}{}^{i} = 0.$$

2. Projective curvature tensor

S. Kawaguchi [5] has defined the projective curvature tensor W^{i}_{jkl} in a special Kawaguchi space as follows:

(2.1)
$$W^{i}_{jkl} = R_{jkl}{}^{i} - \frac{1}{n-1}[\delta^{i}_{j}R_{akl}{}^{a} - \delta^{i}_{k}R_{ajl}{}^{a}] - \frac{x'^{i}}{n+1}\nabla'_{l}R_{jka}{}^{a} +$$
$$+ \frac{x'^{h}}{n^{2}-1}[\delta^{i}_{j}\nabla'_{l}R_{hka}{}^{a} - \delta^{i}_{k}\nabla'_{l}R_{hja}{}^{a}] - \frac{1}{n^{2}-1}[\delta^{i}_{j}R_{kla}{}^{a} - \delta^{i}_{k}R_{jla}{}^{a}] - \frac{1}{n+1}\delta^{i}_{l}R_{jka}{}^{a}.$$

Thus defined projective curvature tensor is a homogeneous function of degree zero with respect to x'^{i} and satisfies the following relations:

(2.2) $$W^{a}_{akl} = W^{a}_{kal} = W^{a}_{kla} = 0,$$

(2.3) $$\nabla'_{a}W^{a}_{jkl} = 0,$$

(2.4) $$W^{i}_{jkl} + W^{i}_{kjl} = 0, \quad W^{i}_{jkl} + W^{i}_{klj} + W^{i}_{ljk} = 0,$$

(2.5) $$W^{i}_{jkl}x'^{l} = W^{i}_{jk}, \quad \nabla'_{l}W^{i}_{jk} = W^{i}_{jkl},$$

where

(2.6)
$$W^{i}_{jk} = K_{jk}{}^{i} + \frac{x'^{i}}{n+1}[\nabla'_{j}K_{ka}{}^{a} - \nabla'_{k}K_{ja}{}^{a}] + \frac{\delta^{i}_{j}}{n+1}\left[K_{ka}{}^{a} + \frac{1}{n-1}\nabla'_{k}(K_{ab}{}^{a}x'^{b})\right] -$$
$$\frac{\delta^{i}_{k}}{n+1}\left[K_{ja}{}^{a} + \frac{1}{n-1}\nabla'_{j}(K_{ab}{}^{a}x'^{b})\right].$$

3. Projectively symmetric special Kawaguchi space

We call an n-dimensional special Kawaguchi space a projectively symmetric special Kawaguchi space if the projective curvature tensor W^i_{jkl} defined by (2.1) satisfies the relation

$$\nabla_m W^i_{jkl} = 0. \tag{3.1}$$

For brevity, we denote such a space by PSK_n-space. Applying ∇_m to (2.1) and using the commutation formula (1.6) and the fact that $\nabla_m x'^i=0$, we find that the condition for a PSK_n-space becomes

$$\begin{aligned}\nabla_m R_{jkl}{}^{i} - \frac{1}{n-1}[\delta^i_j \nabla_m R_{akl}{}^{a} - \delta^i_k \nabla_m R_{ajl}{}^{a}] -\\ - \frac{x'^i}{n+1}[\nabla'_l \nabla_m R_{jka}{}^{a} + B_{mlj}{}^{r} R_{rka}{}^{a} + B_{mlk}{}^{r} R_{jra}{}^{a}] +\\ + \frac{x'^h}{n^2-1}[\delta^i_j\{\nabla'_l \nabla_m R_{hka}{}^{a} + B_{mlk}{}^{r} R_{hra}{}^{a}\} - \delta^i_k\{\nabla'_l \nabla_m R_{hja}{}^{a} + B_{mlj}{}^{r} R_{hra}{}^{a}\}] -\\ - \frac{1}{n^2-1}[\delta^i_j \nabla_m R_{kla}{}^{a} - \delta^i_k \nabla_m R_{jla}{}^{a}] - \frac{\delta^i_l}{n+1}\nabla_m R_{jka}{}^{a} = 0.\end{aligned} \tag{3.2}$$

If the space is recurrent, we have

$$\nabla_m R_{jkl}{}^{i} = V_m R_{jkl}{}^{i}, \tag{3.3}$$

where V_m is a non-zero vector field. If $B_{jkl}{}^{i}=0$, then V_m is independent of x'^i. Letting $B_{jkl}{}^{i}=0$ and using (3.3) in (3.2), we have

$$V_m W^i_{jkl} = 0. \tag{3.4}$$

Since V_m is a non-zero vector field, we have

$$W^i_{jkl} = 0, \tag{3.5}$$

i.e. the space is projectively flat. Thus we have

Theorem 3.1. *If $B_{jkl}{}^{i}=0$, then every recurrent PSK_n-space is projectively flat.*

Now contracting (3.2) with respect to the indices i and m, we have

$$\begin{aligned}\nabla_i R_{jkl}{}^{i} - \frac{1}{n-1}[\nabla_j R_{akl}{}^{a} - \nabla_k R_{ajl}{}^{a}] - \frac{x'^i}{n+1}\nabla'_l \nabla_i R_{jka}{}^{a} +\\ + \frac{x'^h}{n^2-1}[\nabla'_l \nabla_j R_{hka}{}^{a} - \nabla'_l \nabla_k R_{hja}{}^{a}] - \frac{1}{n^2-1}[\nabla_j R_{kla}{}^{a} - \nabla_k R_{jla}{}^{a}] - \frac{1}{n+1}\nabla_l R_{jka}{}^{a} = 0.\end{aligned} \tag{3.6}$$

Transvecting (3.6) with x'^l, we have

$$\nabla_i K_{jk}{}^{i} - \frac{1}{n-1}[\nabla_j K_{ak}{}^{a} - \nabla_k K_{aj}{}^{a}] - \frac{x'^l}{n^2-1}[\nabla_j R_{kla}{}^{a} - \nabla_k R_{jla}{}^{a}] - \frac{x'^l}{n+1}\nabla_l R_{jka}{}^{a} = 0, \tag{3.7}$$

where we have used (1.11) and the fact that $R_{jkl}^{\cdot\cdot\cdot i}$ is homogeneous of degree zero with respect to x'^i. Hence we have

Theorem 3.2. *In a PSK_n-space the relation* (3.7) *holds good.*

Let us now consider the projective transformation of the function $\Gamma^i(x, x')$ defined by the equation

$$\bar{\Gamma}^i(x, x') = \Gamma^i(x, x') + \alpha(x, x')x'^i, \tag{3.8}$$

where $\alpha(x, x')$ is an arbitrary scalar function positively homogeneous of the first degree with respect to x'^i. The tensor W^i_{jkl} is invariant under the projective change (3.8), i.e.,

$$\overline{W}^i_{jkl} = W^i_{jkl}. \tag{3.9}$$

Differentiating (3.9) covariantly and noting the homogeneity property of W^i_{jkl}, we have

$$\begin{aligned}\overline{\nabla}_m \overline{W}^i_{jkl} = \nabla_m W^i_{jkl} - \alpha \nabla'_m W^i_{jkl} + W^a_{jkl}\alpha_{(a)}\delta^i_m + \\ + \alpha_{(a)(m)} W^a_{jkl} x'^i - \alpha_{(j)} W^i_{mkl} - W^i_{akl}\alpha_{(j)(m)}x'^a - W^i_{jml}\alpha_{(k)} - 2\alpha_{(m)} W^i_{jkl} - W^i_{jal}\alpha_{(k)(m)}x'^a - \\ - W^i_{jkm}\alpha_{(l)} - W^i_{jka}\alpha_{(l)(m)}x'^a,\end{aligned} \tag{3.10}$$

where $\overline{\nabla}_m$ denotes covariant differentiation with respect to $\bar{\Gamma}^i$. Contracting (3.10) with respect to the indices i and m and using (2.2), (2.3) and the homogeneity property of $\alpha(x, x')$, we have

$$\begin{aligned}\overline{\nabla}_i \overline{W}^i_{jkl} = \nabla_i W^i_{jkl} + (n-2)\alpha_{(a)} W^a_{jkl} - \\ - x'^a[W^i_{akl}\alpha_{(j)(i)} + W^i_{jal}\alpha_{(k)(i)} + W^i_{jka}\alpha_{(l)(i)}].\end{aligned} \tag{3.11}$$

We now assume that the space and its projective map are both projectively symmetric. Then we have

$$\nabla_m W^i_{jkl} = 0 \quad \text{and} \quad \overline{\nabla}_m \overline{W}^i_{jkl} = 0. \tag{3.12}$$

Consequently, from (3.11), we have

$$(n-2)\alpha_{(a)} W^a_{jkl} - x'^a[W^i_{akl}\alpha_{(j)(i)} + W^i_{jal}\alpha_{(k)(i)} + W^i_{jka}\alpha_{(l)(i)}] = 0. \tag{3.13}$$

We can, therefore, state the following theorem:

Theorem 3.3. *A necessary condition that the projective map of a PSK_n-space be a space of the same kind is that* (3.13) *holds.*

Applying the commutation formula (1.6) to W^i_{lm}, we have

$$(\nabla_j \nabla'_k - \nabla'_k \nabla_j) W^i_{lm} = -B_{jkr}^{\cdot\cdot\cdot i} W^r_{lm} + B_{jkl}^{\cdot\cdot\cdot r} W^i_{rm} + B_{jkm}^{\cdot\cdot\cdot r} W^i_{lr}. \tag{3.14}$$

$^r W^i_{lr}$.

In view of equations (2.5), and (3.1), (3.14) becomes

In view of equations (2.5), and (3.1), (3.14) becomes

$$W^i_{rm} B_{jkl}^{\cdot\cdot\cdot r} + W^i_{lr} B_{jkm}^{\cdot\cdot\cdot r} - W^r_{lm} B_{jkr}^{\cdot\cdot\cdot i} = 0. \tag{3.15}$$

Thus, we have

Theorem 3.4. *In a PSK_n-space, the relation* (3.15) *holds.*

The Bianchi identities for the projective curvature tensor in a special Kawaguchi space have been obtained by one of the present authors [6]. They are given by

$$
\begin{aligned}
(3.16)\qquad & \nabla_m W^i_{jkl}+\nabla_j W^i_{kml}+\nabla_k W^i_{mjl} = \nabla_m R_{jkl}{}^{i}+\nabla_j R_{kml}{}^{i}+\\
&+\nabla_k R_{mjl}{}^{i}-\frac{1}{n-1}[\delta^i_j(\nabla_m R_{akl}{}^{a}-\nabla_k R_{aml}{}^{a})+\\
&+\delta^i_k(\nabla_j R_{aml}{}^{a}-\nabla_m R_{ajl}{}^{a})+\delta^i_m(\nabla_k R_{ajl}{}^{a}-\nabla_j R_{akl}{}^{a})]-\\
&-\frac{x'^i}{n+1}[\nabla'_l(\nabla_m R_{jka}{}^{a}+\nabla_j R_{kma}{}^{a}+\nabla_k R_{mja}{}^{a})]+\\
&+\frac{x'^h}{n^2-1}[\delta^i_j\nabla'_l(\nabla_m R_{hka}{}^{a}-\nabla_k R_{hma}{}^{a})+\delta^i_k\nabla'_l(\nabla_j R_{hma}{}^{a}-\nabla_m R_{hja}{}^{a})+\\
&+\delta^i_m\nabla'_l(\nabla_k R_{hja}{}^{a}-\nabla_j R_{hka}{}^{a})]-\frac{1}{n^2-1}[\delta^i_j(\nabla_m R_{kla}{}^{a}-\nabla_k R_{mla}{}^{a})+\\
&+\delta^i_k(\nabla_j R_{mla}{}^{a}-\nabla_m R_{jla}{}^{a})+\delta^i_m(\nabla_k R_{jla}{}^{a}-\nabla_j R_{kla}{}^{a})]-\\
&-\frac{\delta^i_l}{n+1}(\nabla_m R_{jka}{}^{a}+\nabla_j R_{kma}{}^{a}+\nabla_k R_{mja}{}^{a}).
\end{aligned}
$$

Tansvecting (3.16) with x'^l and using (1.11), (2.5), (1.13) and the fact that $\nabla_m R_{jkl}{}^{i}$ is homogeneous of degree zero with respect to x'^i, we have

$$
\begin{aligned}
(3.17)\qquad & \nabla_m W^i_{jk}+\nabla_j W^i_{km}+\nabla_k W^i_{mj} = -\frac{1}{n-1}[\delta^i_j(\nabla_m K_{ak}{}^{a}-\nabla_k K_{am}{}^{a})+\\
&+\delta^i_k(\nabla_j K_{am}{}^{a}-\nabla_m K_{aj}{}^{a})+\delta^i_m(\nabla_k K_{aj}{}^{a}-\nabla_j K_{ak}{}^{a})]-\\
&-\frac{x'^l}{n^2-1}[\delta^i_j(\nabla_m R_{kla}{}^{a}-\nabla_k R_{mla}{}^{a})+\delta^i_k(\nabla_j R_{mla}{}^{a}-\nabla_m R_{jla}{}^{a})+\\
&+\delta^i_m(\nabla_k R_{jla}{}^{a}-\nabla_j R_{kla}{}^{a})]-\frac{x'^i}{n+1}(\nabla_m R_{jka}{}^{a}+\nabla_j R_{kma}{}^{a}+\nabla_k R_{mja}{}^{a}).
\end{aligned}
$$

For a PSK_n-space, the left hand side of (3.17) is zero and accordingly, we have

Theorem 3.5. *In a PSK_n-space, the relation*

$$
\begin{aligned}
(3.18)\quad & \frac{1}{n-1}[\delta^i_j(\nabla_m K_{ak}{}^{a}-\nabla_k K_{am}{}^{a})+\delta^i_k(\nabla_j K_{am}{}^{a}-\nabla_m K_{aj}{}^{a})+\delta^i_m(\nabla_k K_{aj}{}^{a}-\nabla_j K_{ak}{}^{a})]+\\
&+\frac{x'^l}{n^2-1}[\delta^i_j(\nabla_m R_{kla}{}^{a}-\nabla_k R_{mla}{}^{a})+\delta^i_k(\nabla_j R_{mla}{}^{a}-\nabla_m R_{jla}{}^{a})+\delta^i_m(\nabla_k R_{jla}{}^{a}-\nabla_j R_{kla}{}^{a})]-\\
&-\frac{x'^i}{n+1}(\nabla_m R_{jka}{}^{a}+\nabla_j R_{kma}{}^{a}+\nabla_k R_{mja}{}^{a}) = 0,
\end{aligned}
$$

holds.

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DEPARTMENT OF MATHEMATICS
GORAKHPUR UNIVERSITY
GORAKHPUR (U.P.)
INDIA

AND

DEPARTMENT OF MATHEMATICS,
JABALPUR UNIVERSITY
JABALPUR (M.P.)
INDIA.

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