Atomistic orthomodular lattices and a generalized probability theory

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Introduction

In this paper we try to give a generalized probability theory. More than thirty years ago A. N. KOLMOGOROV has composed the axiomatic theory of probability. Since then many theories have born in order to generalize the theory of Kolmogorov. Some of these wanted to satisfy practical needs. We think first of all the problems, which were raised in quantum theory. For instance A. RÉNYI [7] has given a conditional probability system, in which the conditional probability is the fundamental concept.

Our study is connected with the research of algebraic properties of the proposition lattice associated with a quantum physical experiment started by J. VON NEUMANN [1].

It is known that in quantum theory the set of events associated with an experiment is not a Boolean algebra, but an orthomodular lattice. As we know, the orthomodular lattice is the least general lattice, which is sutiable for the discription of all physical propositional calculus.

V. VARADARAJAN, S. GUDDER and other authors have taken this point of view into consideration, and gave a generalized probability theory [2], [10]. Without reviewing their results in detail, we mention that they set out from the following conditions:

(1) Let an orthomodular σ-lattice L be given, calling its elements events. (2) Let m be a probability measure on L. (3) An $x: B \to x(B)$-homomorphism of the σ-algebra $\mathscr{B}(R^1)$ of Borel sets of the real line into $\mathscr{L}$ is called observable. (It is clear that this concept may be regarded as a generalization of the random variable in the classical probability theory.) (4) The expectation or average value of an observable x is $E(x) = \int \lambda m[x(d\lambda)]$. For the details we refer the reader to [2].

Our system have (1) and (2) in common with the above system; we define the set of events as an orthomodular atomistic σ-lattice $\mathscr{L}$, and interpret a probability measure m on $\mathscr{L}$. Since $\mathscr{L}$ is atomistic, there is a simple way to define the random variable. Denote $\Omega(\mathscr{L})$ the atom space of $\mathscr{L}$. We say that a map x from $\Omega(\mathscr{L})$ into the real line R^1 is a random variable if $\vee x^{-1}(B) \in \mathscr{L}$ for every B Borel set of R^1.
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It is an important fact that in this manner the simple functions (sum, product, ...) of random variables are random variables too.

In connection with $\mathscr{L}$ we mention that the collection of all closed subspaces of a separable Hilbert space H is an orthomodular atomistic σ-lattice. The order of this lattice is the inclusion and the complement of a subspace is defined as its orthocomplement. This example gives us the usual representation of classical quantum mechanics, and so it is the most important example for an orthomodular lattice.

Our paper consists of five chapters:

1. Basic concepts, 2. System of events, 3. Random variables, 4. Probability measures and distributions, 5. The different kinds of the laws of large numbers.

1. Basic concepts

The lattice theoretical concept we use can be found in [6]. Let in this paragraph $\mathscr{L}$ be a lattice. We shall employ the following notations:
The partial ordering of $\mathscr{L}$ will be denoted by $\leqq$. Let S be an arbitrary subset of $\mathscr{L}$. The *least upper bound* and the *greatest lower bound* of the elements of S, if they exist, are denoted by $\sup S=\bigvee S$ and $\inf S=\bigwedge S$ respectively. As usual, the *least element* and the *greatest element* of $\mathscr{L}$ will be denoted by 0 and 1 respectively, provided that 0 and 1 exist. If $\mathscr{L}$ is orthocomplemented, then $a^{\perp}$ denotes the *orthocomplement* of $a\in\mathscr{L}$.

We say that $a, b\in\mathscr{L}$ are *disjoint* or *orthogonal* and we write $a\perp b$ if $a\leqq b^{\perp}$. In an orthocomplemented lattice $\mathscr{L}$ an $S\subseteqq\mathscr{L}$ subset is said to be *orthogonal* if $s_1\perp s_2$ for all $s_1, s_2\in S$.

Later we shall employ the following well known lemmas:

Lemma 1.1 ([6] pp. 132, Theorem 29.13) *Let $\mathscr{L}$ be an orthocomplemented lattice. Then the following statements are equivalent:*

(α) *$\mathscr{L}$ is orthomodular*

(β) *If $a\leqq b$, then $b=a\vee(b\wedge a^{\perp})$.*

(γ) *If $a\leqq b$, then there exists $c\in\mathscr{L}$ such that $a\perp c$ and $a\vee c=b$.*

Note that in (γ) c is unique and $c=b\wedge a^{\perp}=b-a$.

Let a, b be elements of an orthocomplemented lattice $\mathscr{L}$. Then we say that a commutes with b and we write aCb when $a=(a\wedge b)\vee(a\wedge b^{\perp})$.

Lemma 1.2. ([6] pp. 166, Lemma 36.3.) *Let $\mathscr{L}$ be an orthomodular lattice and $a, b\in\mathscr{L}$. Then*

$$(\alpha)\qquad aCb \Leftrightarrow bCa \Leftrightarrow bCa^{\perp} \Leftrightarrow a^{\perp}Cb \Leftrightarrow a^{\perp}Cb^{\perp} \Leftrightarrow b^{\perp}Ca^{\perp} \Leftrightarrow b^{\perp}Ca \Leftrightarrow aCb^{\perp}.$$

(β) *If $a\leqq b$, then aCb.*

(γ) *$a\perp b$ if and only if $a\wedge b=0$ and aCb.*

Lemma 1.3. ([6] pp. 167, Lemma 36.6.) *If $\mathscr{L}$ is an orthomodular lattice and $a\leqq b$, $b-a=b\wedge a^{\perp}=0$, then $a=b$.*

Lemma 1.4. ([6] pp. 167, Lemma 36.7.) *Let a, b and c be three elements of an orthomodular lattice $\mathscr{L}$. If some one of these three elements commutes with the other two, then $\{a, b, c\}$ is a distributive triple, that is the distributive laws hold for all permuta tions of a, b and c.*

If $\{a, b, c\}$ is a distributive triple, we write $(a, b, c)T$. In an atomistic lattice $\mathscr{L}$ we call the set of all atoms of $\mathscr{L}$ the *atom space* of $\mathscr{L}$ and denote it by $\Omega(\mathscr{L})$. If $a \in \mathscr{L}$, then denote the set of all $p \leqq a$, $p \in \Omega(\mathscr{L})$ atoms by $\mathscr{A}(a)$.

In our study we shall often concern ourselves with the properties of the probability measures (states) on an orthomodular lattice and the homomorphism of lattices. The precise definitions of these concepts one can find in [2] or [11].

2. System of events

In the classical theory of probability the collection of events forms a σ-Boolean algebra of subsets of a set (space). It is known that this structure is not the absolute one, namely there is a lattice theoretic treatment of probability theory and the two theories are equivalent. About this one can read in the well known book of D. A. KAPPOS [4].

Generally the lattice of all events is not atomistic, but it can be extended to an atomistic Boolean algebra. Assume that $\mathscr{L}$ is an atomic Boolean algebra that is $0 \in \mathscr{L}$ and each element other then 0 includes at least one atom. If $\mathscr{P}(\mathscr{L})$ denotes the Boolean algebra of all subsets of $\Omega(\mathscr{L})$ and $\mathscr{A}(a)=\{q \in \Omega(\mathscr{L}) | q \leqq a\}$, $a \in \mathscr{L}$, then $\mathscr{L}$ is isomorphic to a sublattice of $\mathscr{P}(\mathscr{L})$ by the isomorphism: $\Phi\colon \mathscr{L} \to \mathscr{P}(\mathscr{L})$, $\Phi(x)=\mathscr{A}(x)$, $x \in \mathscr{L}$.

It came to light that in general in an atomistic lattice two atoms are not orthogonal. However, the investigation of the orthogonality is indispensable in our respect.

We remark that the orthogonality is a characteristic property of a Boolean algebra in the following sence:

Theorem 2.1. *Let $\mathscr{L}$ be an orthocomplemented atomistic lattice and let $\Omega(\mathscr{L})$ be orthogonal. Then $\mathscr{L}$ is a Boolean algebra and isomorphic to a sublattice of $\mathscr{P}(\mathscr{L})$ (power set of $\Omega(\mathscr{L})$)*

PROOF. If $A \perp \Omega(\mathscr{L})$, then $A^{\perp}$ denotes the collection of all elements $p \in \Omega(\mathscr{L})$ such that $p \perp q$ for every $q \in A$. It is easy to verify that
$\mathscr{A}(a^{\perp})=\mathscr{A}(a)^{\perp}$ for each $a \in \mathscr{L}$ and
$\mathscr{A}(a \wedge b)=\mathscr{A}(a) \cap (b)$ for every $a, b \in \mathscr{L}$.
In $\mathscr{P}(\mathscr{L})$ let ' denote the complementation. We shall show that the mapping $\Phi(x)= =\mathscr{A}(x)$, $x \in \mathscr{L}$ of $\mathscr{L}$ into $\mathscr{P}(\mathscr{L})$ is a homomorphism. Since $\mathscr{A}(a)=\mathscr{A}(b)$ if and only if $a=b$, this will complete the proof. Thus, it will be sufficient to show that

$$\mathscr{A}(a^{\perp}) = \mathscr{A}(a)' \tag{1}$$

$$\mathscr{A}(a \wedge b) = \mathscr{A}(a) \cap \mathscr{A}(b) \tag{2}$$

$$\mathscr{A}(a \vee b) = \mathscr{A}(a) \cup \mathscr{A}(b). \tag{3}$$

It follows immediately from the assumptions that $\mathscr{A}(a)^{\perp}=\mathscr{A}(a)'$ therefore (1) holds.
On the other hand $\mathscr{A}(a \vee b)'=\mathscr{A}(a^{\perp} \wedge b^{\perp})=\mathscr{A}(a^{\perp}) \cap \mathscr{A}(b^{\perp})=\mathscr{A}(a)' \cap \mathscr{A}(b)'= =(\mathscr{A}(a) \cup \mathscr{A}(b))'$, thus $\mathscr{A}(a \vee b)=\mathscr{A}(a) \cup \mathscr{A}(b)$, that is (3) holds. Thus the theorem is proved.

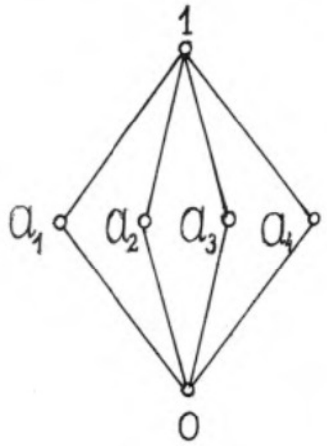
It is obvious that there exists orthocomplemented atomistic lattice $\mathscr{L}$ such that its atom space $\Omega(\mathscr{L})$ is not orthogonal.

Let us consider some examples for such lattices. In the following chapters these examples will be necessary for us.

Example 2.2.

(a)

$\mathscr{L}_1$: , $\Omega(\mathscr{L}_1) = \{a_1, a_2, a_3, a_4\}$.



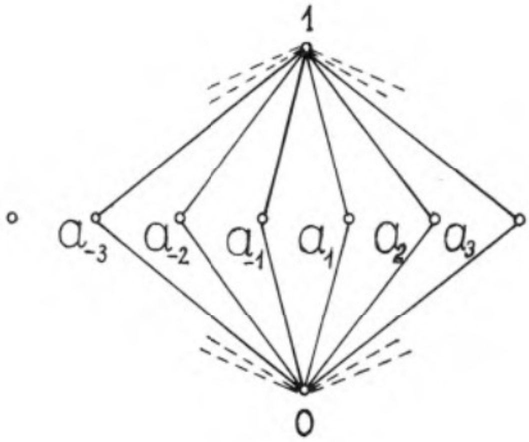
1. ábra

Definition of the orthocomplementation in $\mathscr{L}_1$:

$$a_2 = a_1^{\perp}, \quad a_1 = a_2^{\perp}$$
$$a_4 = a_3^{\perp}, \quad a_3 = a_4^{\perp}$$
$$1 = 0^{\perp}, \quad 0 = 1^{\perp}.$$

Since $a_1 \not\leqq a_3^{\perp} = a_4$, a_1 is not orthogonal to a_3.

(b)



2. ábra

$\mathscr{L}_2$:

$$\Omega(\mathscr{L}_2)=\{a_i,\ i=\pm 1,\ \pm 2,\ \dots\}.$$

Definition of the orthocomplementation in $\mathscr{L}_2$:

$$a_i^{\perp}=a_{-i}\quad (i=\pm 1,\ \pm 2,\ \dots).$$

It is a well known fact that in quantum mechanics two observables are in general not simultaneous measurable. This means that the propositions associated with them can not be verified at the same time. If two propositions (events) a and b may be verified at the same time then we call them *compatible* (written $a\leftrightarrow b$). Mathematically this means that there are mutually disjoint propositions a_1, b_1, c (in the proposition lattice-orthomodular lattice) such that $a=a_1\vee c$, $b=b_1\vee c$.

It is clear that in a Boolean algebra any two elements are compatible. We shall show that in a certain sence this is a characteristic property of a Boolean algebra. (See theorem 2.4.)

The examination of compatibility plays an important role in axiomatic quantum mechanics [10]. In this respect it is an interesting fact that in the atom space of an atomistic orthomodular lattice the orthogonality, compatibility and commutativity (C) are equivalent.

Theorem 2.3. *If $\mathscr{L}$ is an atomistic orthomodular lattice and $p\neq q$; $p, q\in\Omega(\mathscr{L})$, then the following statements are equivalent:* (a) $p\perp q$. (b) $p\leftrightarrow q$. (c) pCq.

PROOF. If $p\perp q$, then $p=p\vee 0$ and $q=q\vee 0$, so $p\leftrightarrow q$, that is (a)$\Rightarrow$(b).

If $p\leftrightarrow q$, then $p\perp q-(p\wedge q)=q$, thus (b)$\Rightarrow$(a).

If $p\perp q$, then $p\wedge q=0$ and $p=p\wedge q^{\perp}$, thus $p=(p\wedge q)\vee(p\wedge q^{\perp})$, so we have pCq, that is (a)$\Rightarrow$(c).

Finally, if pCq, then $p=(p\wedge q)\vee(p\wedge q^{\perp})=p\wedge q^{\perp}\leqq q^{\perp}$, thus $p\perp q$; hence (c)$\Rightarrow$(a).

Theorem 2.4. *An arbitrary atomistic orthomodular lattice $\mathscr{L}$ is a Boolean algebra if and only if its elements are mutually compatible.*

PROOF. In virtue of Theorem 2.3. if the elements of $\mathscr{L}$ are mutually compatible then $\Omega(\mathscr{L})$ is an orthogonal set, thus by Theorem 2.1. $\mathscr{L}$ is a Boolean algebra.

For completeness we prove the inverse direction too. If $\mathscr{L}$ is a Boolean algebra and $a, b\in\mathscr{L}$, then $a-(a\wedge b)=a\wedge(a\wedge b)^{\perp}=a\wedge(a^{\perp}\vee b^{\perp})=b^{\perp}\wedge a\leqq b^{\perp}$, so $a-(a\wedge b)\perp b$. But in an orthomodular lattice $c\leftrightarrow d$ if and only if $c-(c\wedge d)\perp d$ ([2] pp. 73. Corollary 4.3.), thus $a\leftrightarrow b$. This completes the proof.

3. Random variable

Let $(\Omega, \mathscr{A}, \mathscr{P})$ be a probability space and let $\mathscr{B}(R^1)$ be the σ-field of all Borel subsets of the real line R^1. If x is a random variable on $(\Omega, \mathscr{A}, \mathscr{P})$, then $x^{-1}(B)$, $B\in\mathscr{B}(R^1)$ is a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{A}$. Conversely, if h is a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{A}$, then there exists a random variable x on $(\Omega, \mathscr{A}, \mathscr{P})$ which satisfies the equation $x^{-1}(B)=h(B)$ for every $B\in\mathscr{B}(R^1)$. x is unique in the following sence: if y is any other random variable with these properties and $A=\{\omega\in\Omega\,|\,x(\omega)\neq y(\omega)\}$, then $h(A)=0$.

This allows one to define the random variable on $(\Omega, \mathscr{A}, \mathscr{P})$ as a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{A}$. This method is employed by many authors, but it is not always comfortable.

In the introduction we refered to the concept of the random variable applied in axiomatic quantum mechanics. This was examined by V. VARADARAJAN and S. GUDDER (see [10], [2]). The main difficulty in their theory is the study of the algebraic and topological structure of the space of all random variables ($=\sigma$-homomorphisms of $\mathscr{B}(R^1)$ into an orthomodular lattice)
Let now $\mathscr{L}$ be an atomistic orthomodular lattice. We remark that in the following (throughout the chapters 3—5) $\mathscr{L}$ denotes an atomistic orthomodular lattice unless we say something else.

By a real-valued *random variable* on $\mathscr{L}$ we understand a map f from $\Omega(\mathscr{L})$ into the real line R^1 such that $\sup \{p\in\Omega(\mathscr{L})|\, f(p)\in B\}\in\mathscr{L}$ for all $B\in\mathscr{B}(R^1)$.

Let V denote the class of all random variables on $\mathscr{L}$. It is a natural question whether the map $\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$ generated by $f\in V$ will always be a σ-homomorphism or not.

Let O denote the collection of all $f\in V$ for which $\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$ a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{L}$. A counter-example shows that in general $O\neq V$.

Theorem 3.1. *If $\mathscr{L}$ is the lattice $\mathscr{L}_1$ of example* 2.2. (a), *then there exists $f\in V$ such that $f\notin O$.*

PROOF. Let us define f as follows: $f(a_i)=i$ $(i=1, 2, 3, 4)$. Since $\sup f^{-1}(\{i\})=a_i$ and $a_1\perp a_3$, $\sup f^{-1}(\{1\})\perp \sup f^{-1}(\{3\})$ but $\{1\}\cap\{3\}=\emptyset$. This is a contradiction if $\sup f^{-1}(B)$, $B\in\mathscr{B}(R^1)$ a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{L}$, thus $f\notin O$.

We have seen that if $\Omega(\mathscr{L})$ is orthogonal, then $\mathscr{L}$ is a Boolean algebra. We shall prove that in this case $O=V$. (Theorem 3.2.) If $\mathscr{L}$ is an antilattice, then we can write down the random variables $f\in V$ for which $f\in O$. (Theorem 3.5).

An *antilattice* is a complemented lattice in which the supremum of any two nonzero elements is 1.

Theorem 3.2. *If $\Omega(\mathscr{L})$ is orthogonal, then $O=V$.*

PROOF.
a.) $\sup f^{-1}(R^1)=1$, $\sup f^{-1}(\emptyset)=0$ hold for all $f\in V$.
b.) If $Bi\cap Bj=\emptyset$; $B_i, B_j\in\mathscr{B}(R^1)$, then

$$\sup f^{-1}(B_i)\perp \sup f^{-1}(B_j).$$

Namely, it is known (see [5] pp. 598. Lemma 1.2.) that if $b\in\mathscr{L}$, $A\subseteq\mathscr{L}$ and $b\perp a$ for all $a\in A$, then $b\perp\sup A$. Since $\mathscr{L}$ is atomistic, this implies

$$\sup f^{-1}(B_i)\perp \sup f^{-1}(B_j).$$

c.) If $B_1, B_2, \ldots\in\mathscr{B}(R^1)$, then $A=\sup f^{-1}(\bigcup_n B_n)=\bigvee_n \sup f^{-1}(B_n)=C$. $A\geqq C$, because $A=\sup f^{-1}(\bigcup_n B_n)=\sup\bigcup_n f^{-1}(B_n)\geqq\sup f^{-1}(B_n)$ $(n=1, 2, 3, \ldots)$. Hence $A\geqq\bigvee_n \sup f^{-1}(B_n)=C$. Conversely, $A\leqq C$, because by $C=\bigvee_n \sup f^{-1}(B_n)$ $C\geqq p$ for all $p\in f^{-1}(B_n)$, so we have $C\geqq\sup\bigcup_n f^{-1}(B_n)=\sup f^{-1}(\bigcup_n B_n)$.
This means that $A\geqq C$, $A\leqq C$, that is $A=C$.

Theorem 3.3. *If $\mathscr{L}$ is an atomistic Boolean algebra, then $O=V$.*

Proof. This immediately follows from the Theorem 3.2. and the fact that in $\mathscr{L}$ any two atoms are orthogonal.

Returning now to Theorem 3.2., we can see that the orthogonality of $\Omega(\mathscr{L})$ was not used in the proof of c.) part, so the statement c.) is also true without it. Thus the following theorem is valid:

Theorem 3.4. *If $f\in V$, then for arbitrary $E_1, E_2, \ldots \in\mathscr{B}(R^1)$*

$$\sup f^{-1}(\bigcup_n E_n) = \bigvee_n \sup f^{-1}(E_n). \tag{1}$$

If moreover $f\in O$, then

$$\sup f^{-1}(\bigcap_n E_n) = \bigwedge_n \sup f^{-1}(E_n). \tag{2}$$

Theorem 3.5. *Let $\mathscr{L}$ be an antilattice having at least five elements. Then for arbitrary $f\in V$ $f\in O$ if and only if there exists $\alpha\in R^1$ such that $f(p)=\alpha$ for all $p\in\Omega(\mathscr{L})$.*

Proof. I. If $f\in V$ and $f(p)=\alpha$ for all $p\in\Omega(\mathscr{L})$, then for arbitrary $B\in\mathscr{B}(R^1)$

$$\sup f^{-1}(B) = \begin{cases} 1, & \text{if } \alpha\in B \\ 0, & \text{if } \alpha\notin B \end{cases}$$

This means that $x(B)=\sup f^{-1}(B)$ is a σ-homomorphism of $\mathscr{B}(R^1)$ into $\mathscr{L}$, i.e. $f\in O$.

II. If $f\in O$, then

a.) if $f(p)\neq f(q)$ for any two elements $p, q\in\Omega(\mathscr{L})$, then $\sup f^{-1}\{f(p), f(q)\}\perp \perp \sup f^{-1}\{f(r)\}$, where $r\in\Omega(\mathscr{L})$, $r\neq p, q$, but $\{f(p), f(q)\}\cap\{f(r)\}=\emptyset$, thus $f\notin O$. This is a contradiction, therefore there exist $p, q\in\Omega(\mathscr{L})$ such that $f(p)=f(q)$.

b.) If $f(p)=f(q)\neq f(r)$, $p\neq q$, $p\neq r$, then $\sup f^{-1}\{f(p)\}\perp\sup f^{-1}\{f(r)\}$. This is a contradiction because of $\sup f^{-1}(f(p)\}=1$.

Thus, by a.) and b.) there exists $\alpha\in R^1$ such that $f(p)=\alpha$ for all $p\in\Omega(\mathscr{L})$.

Let x be an arbitrary mapping of $\mathscr{B}(R^1)$ into $\mathscr{L}$. We shall employ the following notations:

$I_v=\{x\,|\,x(B)=\sup f^{-1}(B)$ for one $f\in V$ and for all $B\in\mathscr{B}(R^1)\}$

$H=\{x\,|\,x$ is a σ-homomorphism of $\mathscr{B}(R')$ into $\mathscr{L}\}$.

$I_0=I_v\cap H=\{x\,|\,x(B)=\sup f^{-1}(B)$ for one $f\in O$ and for all $B\in\mathscr{B}(R^1)\}$.

It is clear that $I_o\subseteqq I_v$, $I_o\subseteqq H$. We shall show that these inclusions are proper.

Theorem 3.6. *Generally $I_o\neq I_v$, $I_o\neq H$.*

Proof. I. If $\mathscr{L}$ is a finite antilattice, then an arbitrary $f\colon\Omega(\mathscr{L})\to R^1$ mappings forms a random variable, but among these there exists $g\in V$ such that $g\notin O$. This means that $x=\sup g^{-1}\notin I_o$, i.e. $I_o\neq I_v$.

II. Let $\mathscr{L}$ be as in I. and let $p, q\in\Omega(\mathscr{L})$, $p\perp q$. Then, by Theorem 2.3. $p\leftrightarrow q$. It is easy to verify (see for example [11] pp. 118. Lemma 6.7.), that in this case there exists $x\in H$ such that $p=x(E)$, $q=x(F)$ for one $E, F\in\mathscr{B}(R^1)$. But by Theorem 3.5. $x\notin I_o$, otherwise there exists $\alpha\in R^1$ such that $x(G)=1$, if $\alpha\in G$ and $x(G)=0$, if $\alpha\notin G$. This is a contradiction. Thus the theorem is proved.

Our problem now is to examine that in the classical case, when $\mathscr{L}$ is a Boolean algebra, what can be said about I_o, I_v and H.

Theorem 3.7. *If $\mathscr{L}$ is an atomistic Boolean algebra, then* $I_v=I_o=H$.

PROOF. I. If $\mathscr{L}$ is an atomistic Boolean algebra, then by Theorem 3.3. $O=V$, thus evidently $I_v=I_o$.

II. To prove $I_v=H$ it will be sufficient to show that if $x\in H$, then there exists an uniquely determined $f\in V$ such that $x(B)=\sup f^{-1}(B)$ for all $B\in\mathscr{B}(R^1)$.

f can be constructed as follows: Let $r_1, r_2, \dots$ be the sequence of all rational numbers and $E_n=(-\infty, r_n)$. If $p\in\Omega(\mathscr{L})$, then let $f(p)=\inf\{r_n \mid p\leqq x(E_n)\}$. If $p\in f^{-1}(E_k)$, then $f(p)\in E_k$, $p\leqq x(E_k)$, $p\in\mathscr{A}(x(E_k))$; $(\mathscr{A}(a)=\{p\in\Omega(\mathscr{L}) \mid p\leqq a\})$ and conversely, if $p\leqq x(E_k)$, then $f(p)\in E_k$, $p\in f^{-1}(E_k)$. This implies $f^{-1}(E_k)=\mathscr{A}(x(E_k))$. Since $\sup\mathscr{A}(x(E_k))=x(E_k)$, $\sup f^{-1}(E_k)$ exists and

$$\sup f^{-1}(E_k)=x(E_k)$$

Let $s\in R^1$, then $(-\infty, s)=\bigcup_{r_k<s} E_k$ and $\sup f^{-1}((-\infty, s))=\sup f^{-1}(\bigcup_{r_k<s} E_k)=$ $=\bigvee_{r_k<s}\sup f^{-1}(E_k)=\bigvee_{r_k<s} x(E_k)=x((-\infty, s))$, thus

$$\sup f^{-1}((-\infty, s))=x((-\infty, s)).$$

Let $\mathscr{D}$ denote the collection of all subsets E of $\mathscr{B}(R^1)$ for which $\sup f^{-1}(E)\in\mathscr{L}$, $x(E)=\sup f^{-1}(E)$. We shall prove that $\mathscr{D}$ forms a σ-algebra. Since $(-\infty, s)\in\mathscr{D}$ for all $s\in R^1$, therefore $\mathscr{B}(R^1)\subseteqq\mathscr{D}$ i.e. $\mathscr{D}=\mathscr{B}(R^1)$. Since $x(R^1)=\sup f^{-1}(R^1)$, therefore (i) $R^1\in\mathscr{D}$. (ii) If $E, F\in\mathscr{D}$, then $x(E\cup F)=x(E)\vee x(F)=\sup f^{-1}(E)\vee$ $\vee\sup f^{-1}(F)=\sup f^{-1}(E\cup F)$, i.e. $E\cup F\in\mathscr{D}$. (iii) If $E_1, E_2, \ldots\in\mathscr{D}$, then $x(\bigcup_k E_k)=\bigvee_k x(E_k)=\bigvee_k \sup f^{-1}(E_k)=\sup f^{-1}(\bigcup_k E_k)$ i.e. $\bigcup_k E_k\in\mathscr{D}$. Since $\mathscr{L}$ is a Boolean algebra $\mathscr{A}(x(E'))=\Omega(\mathscr{L})\setminus\mathscr{A}(x(E))=f^{-1}(E')$ and we have $\sup f^{-1}(E')\in\mathscr{L}$ and $\sup f^{-1}(E')=x(E')$ i.e.

$$E'\in\mathscr{D} \text{ if } E\in\mathscr{D}. \tag{3}$$

(iv) Let $E, F\in\mathscr{D}$, then by (3)

$$\begin{aligned} x(E\setminus F)&=x(E\cap F^1)=x(E)\wedge x(F')=x(E)\wedge x(F)^\perp= \\ &=\sup f^{-1}(E)\wedge(\sup f^{-1}(F))^\perp=\{(\sup f^{-1}(E))^\perp\vee\sup f^{-1}(F)\}^\perp= \\ &=\{\sup f^{-1}(E'\cup F)\}^\perp=\sup f^{-1}(E\cap F')=\sup f^{-1}(E\setminus F), \end{aligned}$$

therefore $E\setminus F\in\mathscr{D}$.

Thus $\mathscr{D}$ is a σ-algebra indeed, which was to be proved. This means that $x(E)=$ $=f^{-1}(E)$ for all $E\in\mathscr{B}(R^1)$. Finally, we shall prove the uniqueness of f: Let $g\in V$ and suppose that $\sup f^{-1}(E)=\sup g^{-1}(E)$ for all $E\in\mathscr{D}(R^1)$. Assume that $f(p)\neq$ $\neq g(p)$ for one $p\in\Omega(\mathscr{L})$. Then $p\in f^{-1}(f(p))$, $p\in f^{-1}(g(p))$ and hence $\sup f^{-1}(f(p))\wedge\sup f^{-1}(g(p))\geqq p$. However $\sup f^{-1}(f(p))\wedge\sup g^{-1}(g(p))=$ $=\sup f^{-1}(f(p)\cap g(p))=0$. Since this is a contradiction $f(p)=q(p)$ holds for every $p\in\Omega(\mathscr{L})$, thus the proof of the theorem is complete.

Let $\mathscr{L}$ be an orthomodular σ-lattice. A subset $\mathscr{L}'\subseteqq\mathscr{L}$ is said to be a *Boolean subalgebra* if (i) $\mathscr{L}'$ is a subalgebra of $\mathscr{L}$, (ii) $\mathscr{L}'$ is a Boolean algebra. If a Boolean subalgebra $\mathscr{L}'$ forms a σ-lattice too, then it is called *Boolean sub σ-algebra*. A Boolean algebra $\mathscr{L}$ is said to be *separable* if there exists a countable set $D \subseteqq \mathscr{L}$ such that the smallest sub σ-algebra of $\mathscr{L}$ containing D is $\mathscr{L}$ itself.

Varadarajan has proved (see [11] pp. 123, Lemma 6.16.) that a Boolean sub σ-algebra $\mathscr{L}'$ is separable if and only if there exists a σ-homomorphism of $\mathscr{B}(R^1)$ onto $\mathscr{L}'$. If $\mathscr{L}$ is an atomistic orthomodular σ-lattice and $x \in H$, then from this it is obvious that $\mathscr{R}(x)$ is a Boolean subalgebra of $\mathscr{L}$, where $\mathscr{R}(x)$ is the range of x. In the following theorem we shall show that if $x \in I_o$, then $\mathscr{R}(x)$ is isomorphic to a sublattice of $\mathscr{P}(\mathscr{L})$, the power set of $\Omega(\mathscr{L})$.

Theorem 3.8. *If* $x \in I_o$, *then*

(1) $\mathscr{R}(x)$ *is a Boolean sub* σ*-algebra of* $\mathscr{L}$.

(2) *The map* $\Phi: a \to \mathscr{A}(a)$, $a \in \mathscr{R}(x)$ *is an isomorphism of* $\mathscr{R}(x)$ *onto one sublattice of* $\mathscr{P}(\mathscr{L})$, *i.e.*

(a) $\mathscr{A}(a^\perp) = \Omega(\mathscr{L}) \setminus \mathscr{A}(a)$

(b) $\mathscr{A}(a \wedge b) = \mathscr{A}(a) \cap \mathscr{A}(b)$

(c) $\mathscr{A}(a \vee b) = \mathscr{L}(a) \cup \mathscr{A}(b)$

(d) Φ *is a one-one mapping.*

Proof. To prove (1) it will be sufficient to show that $(a, b, c)T$ for all $a, b, c \in \mathscr{R}(x)$.

I. Let $a, b, c \in \mathscr{R}(x)$ i.e. $a = x(A)$, $b = x(B)$, $c = x(C)$; $A, B, C \in \mathscr{B}(R^1)$. Then we have $(a \vee b) \wedge c = x((A \cup B) \cap C) = x \quad ((A \cap C) \cup (B \cap C)) = x(A \cap C) \vee x(B \cap C) = (a \wedge c) \vee \vee (b \wedge c)$. Thus $(a, b, c)T$.

II. (2)/(b) and (2)/(d) are evident. If $a \in \mathscr{R}(x)$, then $a = x(E)$ for some $E \in \mathscr{B}(R^1)$ and $a^\perp = x(E') = x(R' \setminus E)$. Since $x(E) = \sup f^{-1}(E)$, $f \in O$, therefore $f^{-1}(E) \subseteqq \subseteqq \mathscr{A}(x(E))$. Furthermore $x(E') = \sup f^{-1}(E')$ and we have $\mathscr{A}(x(E')) \supseteqq f^{-1}(E')$. But $\mathscr{A}(x(E')) = \mathscr{A}(x(E)^\perp) = \mathscr{A}(x(E))^\perp \subseteqq \Omega(\mathscr{L}) \setminus (x(E)) = f^{-1}(E')$, where $\mathscr{A}(y)^\perp = = \{p \in \Omega(\mathscr{L}) \mid p \perp q$ for all $q \in \mathscr{A}(y)\}$. Then $\mathscr{A}(x(E')) = f^{-1}(E')$, $\mathscr{A}(x(E)) = f^{-1}(E)$, thus (2)/(a) holds.

If $a, b \in \mathscr{R}(x)$, then $\mathscr{A}(a \vee b) = \mathscr{A}(x(E) \vee x(F)) = \mathscr{A}(x(E \cup F)) = f^{-1}(E \cup F) = = f^{-1}(E) \cup f^{-1}(F) = \mathscr{A}(x(E)) \cup \mathscr{A}(x(F)) = \mathscr{A}(a) \cup \mathscr{A}(b)$ i.e. (2)/(c) holds. This completes the proof.

Let $f_1, f_2, \ldots, f_n \in V$ and let $\psi(x_1, \ldots, x_n)$: $R^n \to R^1$ be a real valued function of n variables. Then let $\psi(f_1, \ldots, f_n)$ denote the mapping $\Omega(\mathscr{L}) \to R^1$ such that $\psi(f_1, \ldots, f_n)(p) = \psi(f_1(p), \ldots, f_n(p))$ for every $p \in \Omega(\mathscr{L})$. This is a natural way to define some function of n random variables, but we have to pose the question whether $\psi(f_1, \ldots, f_n)$ is an element of V if $f_1, \ldots, f_n \in V$. The answer depends on ψ and the algebraic properties of $\mathscr{S}(\Omega)$, where $\mathscr{S}(\Omega)$ is the class of all subsets A of $\Omega(\mathscr{L})$ for which $\sup A \in \mathscr{L}$.

Theorem 3.9. *If* $\mathscr{S}(\Omega)$ *is a* σ*-algebra and* $\psi(x_1, \ldots, x_n)$ *is a Borel function, then* $f = \psi(f_1, \ldots, f_n) \in V$ *for every* $f_1, \ldots, f_n \in V$.

Proof. Since $(\Omega(\mathscr{L}), \mathscr{S}(\Omega))$ is a measurable space and $f_1, \ldots, f_n \mathscr{S}(\Omega)$ measurable functions, therefore $f = \psi(f_1, \ldots, f_n)$ is also $\mathscr{S}(\Omega)$ measurable, i.e. $f^{-1}(B) = \Phi(\psi^{-1}(B)) \in \mathscr{S}(\Omega)$ for all $B \in \mathscr{B}(R^1)$, where $\Phi(p) = (f_1(p), f_2(p), \ldots, f_n(p)) \in R^n$, $p \in \Omega(\mathscr{L})$. Thus $f \in V$, which was to be proved.

Theorem 3.10. *If $f_1 f_2 \in V$, $f_1 - f_2 \in V$ hold for any $f_1, f_2 \in V$, then $\mathscr{S}(\Omega)$ is a σ-algebra.*

PROOF. Let $A_1, A_2 \in \mathscr{S}(\Omega)$ and let f_i be the characteristic function of $A_i (i=1, 2)$: $f_i(p)=1$ if $p \in A_i$ and $f_i(p)=0$ if $p \notin A_i$ $(i=1, 2)$. It is plausible that f_1, f_2 are random variables, furthermore $f=f_1 f_2 \in V$, $g=f_1-f_2 \in V$ by assumption. Hence $A_1 \cap A_2 = f^{-1}(1) \in \mathscr{S}(\Omega)$ and $A_1 \setminus A_2 = g^{-1}(1) \in \mathscr{S}(\Omega)$. Since $\mathscr{L}$ forms a σ-lattice, therefore $A_1, A_2, \ldots \in \mathscr{S}(\Omega)$ implies $\sup (\bigcup_i A_i) = \bigvee_i \sup A_i \in \mathscr{L}$ i.e. $\bigcup_i A_i \in \mathscr{S}(\Omega)$. Besides this $\Omega(\mathscr{L}) \in \mathscr{S}(\Omega)$, thus $\mathscr{S}(\Omega)$ is a σ-algebra indeed.

By the Theorems 3.9. and 3.10. we can say that the Borel functions of random variables are themselves random variables if and only if $\mathscr{S}(\Omega)$ is a σ-algebra. This condition holds evidently if $\mathscr{L}$ is complete.

Let us return now to the question under what conditions $O=V$ is valid. In the following theorem we shall give a necessary and sufficient condition for $O=V$.

Theorem 3.11. *If $\mathscr{S}(\Omega)$ forms an algebra, then $O=V$ holds if and only if $\mathscr{L}$ is a Boolean algebra.*

PROOF. It will be sufficient to show that $O=V$ implies that $\mathscr{L}$ is a Boolean algebra, since the converse is true by Theorem 3.3. Let $p, q \in \Omega(\mathscr{L})$, $f(p)=0$ and let $f(r)=1$ for all $r \in \Omega(\mathscr{L}) \setminus p$. Then evidently $f \in V = O$, i.e. $\sup f^{-1}(B)$, $B \in \mathscr{B}(R^1)$ is a σ-homomorphism. Hence $\sup f^{-1}(0) \perp \sup f^{-1}(1)$, thus $p \perp \sup \big(\Omega(\mathscr{L}) \setminus p\big)$. This means that $p \perp q$ for any two elements of $\Omega(\mathscr{L})$, thus by Theorem 2.1. $\mathscr{L}$ is a Boolean algebra.

4. Probability measures and distributions

In the following two chapters $\mathscr{L}$ will denote an atomistic orthomodular σ-lattice again, unless we shall say someone else.

Let μ be a probability measure on $\mathscr{L}$. Gudder has shown (see [2] pp. 98.) that generally μ is not subadditive; that is $\mu(a \vee b) \leqq \mu(a) + \mu(b)$ do not hold for all $a, b \in \mathscr{L}$.

In the conventional theory a random variable ξ always generates a measure $\mathscr{P}_\xi$ on $\mathscr{B}(R^1)$ in the following sence:

$$\mathscr{P}_\xi(E) = \mathscr{P}\big(\xi^{-1}(E)\big), \quad E \in \mathscr{B}(R^1)$$

where ξ is defined over the probability space $(\Omega, \mathscr{F}, \mathscr{P})$. This is not necessary true in our generalized theory, as the following theorem shows.

Theorem 4.1. *Let $f \in V$ and let μ be a probability measure on $\mathscr{L}$. Then $m^*[a, b) = \mu\big(\sup f^{-1}[a, b)\big)$ is a map of the family of all half-intervals $[a, b) \subset R^1$ into $[0, 1] \subset R^1$. In general m^* is non additive.*

PROOF. We shall prove our statement by means of a counter-example.

Let $\mathscr{L}$ be as the lattice $\mathscr{L}_1$ in Example 2.2/(a) and let $\mu(a_i) = 1/2$ $(i=1, 2, 3, 4)$, $\mu(1)=1$, $\mu(0)=0$. Then μ is a probability measure on $\mathscr{L}$. The atom space: $\Omega(\mathscr{L}) =$

$=\{a_1, a_2, a_3, a_4\}$. If $f(a_i)=i$ $(i=1, 2, 3, 4)$, then

$$\mu\left(\sup f^{-1}\left[\frac{1}{2}, \frac{7}{2}\right)\right)=1 \tag{1}$$

$$\mu\left(\sup f^{-1}\left[\frac{1}{2}, \frac{5}{2}\right)\right)=1 \tag{2}$$

$$\mu\left(\sup f^{-1}\left[\frac{5}{2}, \frac{7}{2}\right)\right)=\frac{1}{2}. \tag{3}$$

By (1), (2) and (3)

$$m^*\left(\left[\frac{1}{2}, \frac{7}{2}\right)\right) \neq m^*\left(\left[\frac{1}{2}, \frac{5}{2}\right)\right)+m^*\left(\left[\frac{5}{2}, \frac{7}{2}\right)\right).$$

If $f\in V$, then the function $F(x)=\mu\left(\sup f^{-1}(-\infty, x)\right)$ is called the *distribution function* of f. Ahead of examination of distribution functions we prove a property of the probability measure μ, which is well known in the classical measure theory.

Theorem 4.2. *Let $\mathscr{L}$ be an orthomodular σ-lattice. Then for all probability measures μ on $\mathscr{L}$ hold*

(1) If $a_1\geqq a_2\geqq\dots$; $a_i\in\mathscr{L}$, *then* $\lim\limits_{n\to\infty}\mu(a_n)=\mu(\bigwedge\limits_n a_n)$.

(2) If $b_1\leqq b_2\leqq\dots$; $b_i\in\mathscr{L}$, *then* $\lim\limits_{n\to\infty}\mu(b_n)=\mu(\bigvee\limits_n b_n)$.

Proof. Since $\mathscr{L}$ is orthomodular, (1)⇔(2). Furthermore, if

(3) $\lim\limits_{n\to\infty}\mu(a_n)=0$ for arbitrary $a_1\geqq a_2\geqq\dots$, $\bigwedge\limits_n a_n=0$, then (3) implies (1). Namely, if $a_1\geqq a_2\geqq\dots$ and $\bigwedge\limits_n a_n=a$, then we have

$$a_1-a \geqq a_2-a \geqq\dots \text{ and } \bigwedge_n (a_n-a)=\bigwedge_n (a_n\wedge a^\perp)=a^\perp\wedge(\bigwedge_n a_n)=0.$$

From this we get

$$\lim_{n\to\infty}\mu(a_n-a)=\lim_{n\to\infty}\left(\mu(a_n)-\mu(a)\right)=0, \quad \text{i.e.} \quad \lim_{n\to\infty}\mu(a_n)=\mu(a).$$

Coming to the proof of (3), let $a_1\geqq a_2\geqq\dots$, $\bigwedge\limits_n a_n=0$. We shall prove that

$$a_1=\bigvee_{n=1}^{\infty}(a_n-a_{n+1})=\bigvee_{n=1}^{\infty}(a_n\wedge a_{n+1}^\perp). \tag{4}$$

Evidently $a_1\geqq\bigvee\limits_{n=1}^{\infty}(a_n\wedge a_{n+1}^\perp)=b$.

By Lemma 1.3. (4) holds if $a_1-b=0$. The proof of $a_1-b=0$ is the following: We have

$$a_1-b=a_1-\{(a_1\wedge a_2^\perp)\vee(a_2\vee a_3^\perp)\vee\dots\}=a_1\wedge(a_1^\perp\vee a_2)\wedge(a_2^\perp\vee a_3)\wedge\dots$$

Let us see elements $a_n, a_n^\perp, a_{n+1}$. It is obvious that $a_n C a_{n+1}$, i.e. a_n commutes with a_{n+1}, because $a_n \geqq a_{n+1}$. In a similar manner $a_n^\perp C a_{n+1}$. (see Lemma 1.2.). Hence by Lemma 1.4. we get $(a_n, a_n^\perp, a_{n+1})T$, thus $a_n \wedge (a_n^\perp \vee a_{n+1}) = (a_n \wedge a_n^\perp) \vee (a_n \wedge a_{n+1}) = a_n \wedge a_{n+1}$. By this equality

$$a_1 - b = a_1 \wedge (a_1^\perp \vee a_2) \wedge (a_2^\perp \vee a_3) \wedge \ldots = c_1$$

$$a_1 - b = a_1 \wedge a_2 \wedge (a_2^\perp \vee a_3) \wedge (a_3^\perp \vee a_4) \wedge \ldots = c_2$$

$$a_1 - b = a_1 \wedge a_2 \wedge a_3 \wedge (a_3^\perp \vee a_4) \wedge (a_4^\perp \vee a_5) \wedge \ldots = c_3$$

$$\vdots$$

$$a_1 - b = a_1 \wedge a_2 \wedge \ldots \wedge a_n \wedge (a_n^\perp \vee a_{n+1}) \wedge (a_{n+1}^\perp \vee a_{n+2}) \wedge \ldots = c_n$$

We shall prove that $\bigwedge_{n=1}^{\infty} c_n \leqq \bigwedge_{n=1}^{\infty} a_n$.

We have $c_n \leqq a_1 \wedge a_2 \wedge \ldots \wedge a_n$ $(n = 1, 2, \ldots)$, hence $\bigwedge_n c_n \leqq \bigwedge_n (a_1 \wedge a_2 \wedge \ldots \wedge a_n) = \bigwedge_n a_n$, thus (4) is true. In equality (4) $(a_n - a_{n+1}) \perp (a_m - a_{m+1})$ $(m \neq n)$, because if for example $m < n$, then $a_m \leqq a_{n+1}$ and hence

$$(a_n \wedge a_{n+1}^\perp)^\perp = a_n^\perp \vee a_{n+1} \geqq a_m \wedge a_{m+1}^\perp.$$

Then $\mu(a_1) = \sum_{n=1}^{\infty} \mu(a_n - a_{n+1})$.

Generally, if $b \geqq a$, then $\mu(b-a) = \mu(b) - \mu(a)$ because $b = a \vee (b-a)$ and $a \perp (b-a)$. Then, obviously

$$\mu(a_1) = \sum_{n=1}^{\infty} \mu(a_n - a_{n+1}) = \sum_{n=1}^{\infty} (\mu(a_n) - \mu(a_{n+1})),$$

i.e.

$$\lim_{n \to \infty} \mu(a_n) = \lim_{n \to \infty} \sum_{k=n}^{\infty} \mu(a_k - a_{k+1}) = 0.$$

This completes the proof.

By Theorem 4.2. one can find some elementary properties of distribution function.

Theorem 4.3. *If $F(x)$ is the distribution function of $f \in V$ with respect to the measure μ, then*

(a) *$F(x)$ is monotone increasing.*

(b) *$F(x)$ is continuos from the left.*

(c) *If $f \in V$, then $\lim_{x \to \infty} F(x) = 1$,*

if *$f \in O$, then $\lim_{\to -\infty} F(x) = 0$.*

Proof. (a) If $x > y$, then

$$a = \sup f^{-1}(-\infty, x) \leqq \sup f^{-1}(-\infty, y) = b,$$

thus

$$\mu(a) \leqq \mu(b) \quad \text{i.e.} \quad F(x) \leqq F(y).$$

(b) Let $h_1 \geqq h_2 \geqq \dots$, $h_n > 0$ $(n=1, 2, \dots)$ and $\lim_{n\to\infty} h_n = 0$. Then $(-\infty, x) = \bigcup_n (-\infty, x-h_n)$, thus by Theorem 3.4.

$$\sup f^{-1}(-\infty, x) = \bigvee_n \sup f^{-1}(-\infty, x-h_n) = \bigvee_n a_n,$$

where

$$a_n = \sup f^{-1}(-\infty, x-h_n).$$

Furthermore $a_1 \leqq a_2 \leqq \dots$, so by Theorem 4.2. we have

$$F(x) = \mu(\sup f^{-1}(-\infty, x)) = \lim_{n\to\infty} \mu(a_n) = \lim_{n\to\infty} F(x-h_n).$$

(c) I. Let $x_1, x_2, \dots \in R^1$ with $x_1 \leqq x_2 \leqq \dots$ and $\lim_{n\to\infty} x_n = +\infty$. Then $\bigvee_n \sup f^{-1}(-\infty, x_n) = 1$ i.e.

$$\lim_{n\to\infty} F(x_n) = \mu(1) = 1.$$

II; Let $y_1, y_2, \dots \in R^1$ with $y_1 \geqq y_2 \geqq \dots$ and $\lim_{n\to\infty} y_n = -\infty$. Suppose that $f \in O$. Then by Theorem 3.4. $\bigwedge_n \sup f^{-1}(-\infty, y_n) = \sup f^{-1}(\bigcap_n (-\infty, y_n)) = 0$. Hence $\lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} \mu(\sup f^{-1}(-\infty, y_n)) = \mu(\bigwedge_n \sup f^{-1}(-\infty, y_n))$. Thus the theorem is proved.

Let $F(x)$ be the distribution function of $f \in V$ with respect to the probability measure μ. Then let $\mathscr{P} = \mathscr{P}_f^\mu$ denote the Lebesgue—Stieltjes measure induced by $F(x)$. If $m^*(B) = \mu(\sup f^{-1}(B))$, $B \in \mathscr{B}(R^1)$, then we know that generally m^* is not a measure. However, if $f \in O$, then m^* is a measure on $\mathscr{B}(R^1)$, and for all $B \in \mathscr{B}(R^1)$ $m^*(B) = \mathscr{P}(B)$.

Theorem 4.4. *If $f \in O$ and μ is a probability measure on $\mathscr{L}$, then $m^*(B) = \mu(\sup f^{-1}(B))$, $B \in \mathscr{B}(R^1)$ is a measure on $\mathscr{B}(R^1)$ and for all $B \in \mathscr{B}(R^1)$ $m^*(B) = \mathscr{P}_f^\mu(B)$.*

PROOF. The proof is very simple and so it will not be presented here.

5. The different kinds of the laws of large numbers

In any probability theory the laws of large numbers have theoretical importance. These laws show the relationships between theoretics and practice.

In this chapter we shall give some fundamental concepts (expectation, variance, independence, different types of convergence, ...). Afterward we shall prove a simple form of the laws of large numbers for uncorrelated random variables.

Let $f \in V$ and let $F(x)$ be the distribution function of f with respect to the probability measure μ. Then the Lebesgue—Stieltjes integral $\int x dF(x) = E(f)$ is called the *expectation* of f, provided that this integral exists. The *variance* $D^2(f)$ of f is defined by

$$D^2(f) = E([f-E(f)]^2).$$

The random variables $f_1, f_2, \dots, f_n \in V$ are *independent* with respect to the probability measure μ if

$$\mu\left(\bigwedge_{i=1}^{n} \sup f_i^{-1}(E_i)\right) = \prod_{i=1}^{n} \mu\left(\sup f_i^{-1}(E_i)\right)$$

holds for arbitrary $E_i \in \mathscr{B}(R^1)$, $i=1, 2, \dots, n$. The random variables $f_1, f_2, \dots \in V$ are independent if any n of them are independent for $n=2, 3, \dots$. If $f_1, f_2, \dots, f_n \in V$, then the function

$$\mu\left(\bigwedge_{i=1}^{n} \sup f_i^{-1}(-\infty, x_i)\right) = F_{f_1, f_2, \dots, f_n}(x_1, x_2, \dots, x_n)$$

is called the *joint distribution function* of the random variables $f_1, f_2, \dots, f_n$. It follows from the definition that if $f_1, f_2, \dots, f_n \in V$ are independent, then

$$F_{f_1, f_2, \dots, f_n}(x_1, x_2, \dots, x_n) = F_{f_1}(x_1) F_{f_2}(x_2) \dots F_{f_n}(x_n),$$

where $x_1, x_2, \dots, x_n$ are arbitrary real numbers and $F_{f_i}(x_i)$ is the distribution function of f_i.

The different kinds of the laws of large numbers are based on different types of convergence.

If $a_i \in \mathscr{L}$ $(i=1, 2, \dots)$, then let

$$\limsup a_i = \bigwedge_{j=1}^{\infty} (a_j \vee a_{j+1} \vee \dots).$$

If $f_n, f \in V$ $(n=1, 2, \dots)$ and

$$\limsup \{\sup (f_n - f)^{-1}(R^1 \setminus [-\varepsilon, \varepsilon])\} = 0$$

holds for all $\varepsilon > 0$ real numbers, then we say that f_n converges *everywhere* to f. f_n converges *almost everywhere* (μ) to f if

$$\mu\left(\limsup \left[\sup (f_n - f)^{-1}(R^1 \setminus [-\varepsilon, \varepsilon])\right]\right) = 0 \quad \text{for all} \quad \varepsilon > 0.$$

We say that $f_n \to f$ *in measure* (μ) if

$$\mu\left(\sup (f_n - f)^{-1}(R^1 \setminus [-\varepsilon, \varepsilon])\right) \to 0 (n \to \infty) \quad \text{for all} \quad \varepsilon > 0.$$

Note that if $f_n, f \in O$ $(n=1, 2, \dots)$, then $f_n \to f$ in measure (μ) if and only if

$$\mathscr{P}^{\mu}_{f_n - f}(R^1 \setminus [-\varepsilon, \varepsilon]) \to 0 (n \to \infty) \quad \text{for every} \quad \varepsilon > 0.$$

This follows immediately from the Theorem 4.4. If $f_n, f \in V$ and $\lim_{n \to \infty} E\left((f_n - f)^2\right) = 0$, then we say that $f_n \to f$ *in mean*.

Now we prove a simple but very useful inequality.

Theorem 5.1. *If $\alpha(x)$ is an increasing and strictly positive function on $(0, \infty)$, and $\alpha(x) = \alpha(-x)$, $x \in R^1$, then for arbitrary $f \in V$ such that $E(\alpha(f)) < \infty$ holds*

$$\mathscr{P}^{\mu}_{f}(R^1 \setminus [-\varepsilon, \varepsilon]) \leqq \frac{E\left(\alpha(f)\right)}{\alpha(\varepsilon)} \quad \textit{for each} \quad \varepsilon > 0.$$

PROOF. Evidently

$$E(\alpha(f)) = \int \alpha(x)\, dF_f(x) \geqq \int_{|x|>\varepsilon} \alpha(x)\, dF_f(x) \geqq \alpha(\varepsilon)\mathscr{P}_f(|x| \geqq \varepsilon).$$

By using this inequality easy to get a simple form of the laws of large numbers.

$f_1, f_2, \ldots, f_n \in V$ is said to be *uncorrelated* if $D^2(f_1+f_2+\ldots f_n) = \sum_{i=1}^{n} D^2(f_i)$. $q_1, q_2, \ldots \in V$ is called *uncorrelated* if every finite subsequence of $\{g_1, g_2, \ldots\}$ is uncorrelated. If $f \in V$, then $m^*(B) = \mu(\sup f^{-1}(B))$, $B \in \mathscr{B}(R^1)$ is called the *probability distribution* of f with respect to the probability measure μ. A family of random variables having the same distribution is said to be *identically distributed.*

Theorem 5.2. *If $f_1, f_2, \ldots$ are identically distributed and uncorrelated random variables, then*

$$\mathscr{P}^{\mu}_{\frac{S_n - E(S_n)}{n}}(R^1 \setminus [-\varepsilon, \varepsilon]) \to 0 \quad (n \to \infty)$$

for all $\varepsilon > 0$, where $S_n = f_1 + f_2 + \ldots + f_n$.

PROOF. By Theorem 5.1.

$$\mathscr{P}^{\mu}_{\frac{S_n - E(S_n)}{n}}(R^1 \setminus [-\varepsilon, \varepsilon]) \leqq \frac{E([S_n - E(S_n)]^2)}{\varepsilon^2 n^2} =$$

$$= \frac{D^2(f_1+f_2+\ldots+f_n)}{\varepsilon^2 n^2} = \frac{nD^2(f_1)}{n^2 \varepsilon^2} \to 0 \quad (n \to \infty)$$

which was to be proved.

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*) One can find more detailed bibliography in (2)