Generalized difference-property for functions of several variables

By K. LAJKÓ (Debrecen—Gödöllő)

0. Introduction

Setting out from N. G. DE BRUIJN's investigations (see [2]) numerous authors studied difference-properties (e.g. [2], [3], [4]). An *algebraic difference-property*, first to be found in the paper of Z. DARÓCZY—K. LAJKÓ—L. SZÉKELYHIDI [5], is the following:

Denote by P the set of positive elements of an ordered field and let A be an additive Abelian group. Let $f\colon P\to A$ be an arbitrary function and $\Delta_\lambda f\colon P\to A$ be the function defined by

$$\Delta_\lambda f(x)=f(\lambda x)-f(x)\quad (x\in P) \tag{1}$$

for all fixed $\lambda\in P$. The function $\alpha\colon P\to A$ is called a *Jensen-function* if

$$2\alpha\left(\frac{x_1+x_2}{2}\right)=\alpha(x_1)+\alpha(x_2)\quad (x_1, x_2\in P) \tag{2}$$

holds. Denote by $J(P\to A)$ the class of Jensen-functions.

The next theorem is due to Z. DARÓCZY—K. LAJKÓ—L. SZÉKELYHIDI.

Theorem 0.1. *Let $f\colon P\to A$ be an arbitrary function, such that the function $\Delta_\lambda f$ defined by* (1) *is a Jensen-function for every fixed $\lambda\in P$. Then there exists a function $\alpha\in J\ (P\to A)$ such that the function $m(x)=f(x)-\alpha(x)$ $(x\in P)$ is a homomorphism, i.e.*

$$m(x_1x_2)=m(x_1)+m(x_2)\quad (x_1, x_2\in P) \tag{3}$$

holds.

This result shows that *the Jensen-property is a difference-property.*

In this paper we are going to investigate a generalization of the above difference-property.

Our notations are the following: Denote by P^n the set

$$\overset{1}{\breve{P}}\times\overset{2}{\breve{P}}\times\dots\times\overset{n}{\breve{P}}\quad\text{and for}$$

$\underline{\lambda}=(\lambda_1,\dots,\lambda_n)\in P^n$, $\underline{x}=(x_1,\dots,x_n)\in P^n$ let $\underline{\lambda}\underline{x}\doteq(\lambda_1x_1,\dots,\lambda_nx_n)$.

If $\underline{x}\in P^n$ then $\underline{x}_i(t)$ will denote the vector

$$(x_1,\dots,x_{i-1}, t, x_{i+1},\dots,x_n).$$

Let $F\colon P^n\to A$ be an arbitrary function and $\Delta_{\underline{\lambda}}F\colon P^n\to A$ be the function defined by

$$\Delta_{\underline{\lambda}}F(\underline{x}) = F(\underline{\lambda x})-F(\underline{x}) \quad (\underline{x}, \underline{\lambda}\in P^n). \tag{4}$$

The function $H\colon P^n\to A$ *is said to be a generalized Jensen-function, if*

$$2H\left(\underline{x}_i\left(\frac{t+s}{2}\right)\right) = H(\underline{x}_i(t))+H(\underline{x}_i(s)) \tag{5}$$

holds for every $i=1, \dots, n$; $\underline{x}\in P^n$ *and* $t, s\in P$. *The class of generalized Jensen-functions will be denoted by* $J\ (P^n\to A)$.

The property T *is said to be a generalized difference-property if the following holds: for any* $F\colon P^n\to A$, *such that* $\Delta_{\underline{\lambda}}F\colon P^n\to A$ *has the property* T *for every* $\underline{\lambda}\in P^n$ *the representation*

$$F(\underline{x}) = H^*(\underline{x})+\sum_{i=1}^{n} m_i(x_i) \quad (\underline{x}\in P^n) \tag{6}$$

is valid, where H^* *is a function of property* T *and* $m_i\colon P\to A$ *are homomorphisms of* P *into* A $(i=1, \dots, n)$.

We prove that *the generalized Jensen-property is a generalized difference-property in this general sense.* Besides in the case $\mathbf{R}_+=\{x|x>0\}$ we shall give the general form of generalized Jensen-functions.

1. A generalized difference-property

The generalization of theorem 0.1. can be formulated as follows:

Theorem 1.1. *Let* $F\colon P^n\to A$ *be such a function that* $\Delta_{\underline{\lambda}}F\in J(P^n\to A)$ *for all* $\underline{\lambda}\in P^n$, *then*

$$F(\underline{x}) = \alpha(\underline{x})+\sum_{i=1}^{n} m_i(x_i) \quad (\underline{x}\in P^n), \tag{7}$$

where $\alpha\in J\ (P^n\to A)$ *and* $m_i\colon P\to A$ $(i=1, \dots, n)$ *are homomorphisms of* P *into* A.

Proof. Our method is induction on n, the number of variables of F.

a) *If* $n=1$, the proposition is just the theorem 0.1. which was proved in [5].

b) *Let us suppose that the theorem holds for functions of n variables*, then we show that it is true for functions of $n+1$ variables.

We shall use the notations

$$(\underline{x}, x_{n+1}) = (x_1, \dots, x_n, x_{n+1}),$$
$$(\underline{\lambda}, \lambda_{n+1}) = (\lambda_1, \dots, \lambda_n, \lambda_{n+1})$$

for elements of P^{n+1}.

Assume that $\Delta_{(\underline{\lambda}, \lambda_{n+1})}F\in J(P^{n+1}\to A)$.

(i) *First we demonstrate* that the function defined by

$$\varphi(\underline{x}, x_{n+1}) = F(\underline{x}, 2x_{n+1})-F(\underline{x}, x_{n+1}), \quad ((\underline{x}, x_{n+1})\in P^{n+1}) \tag{9}$$

has the form

(10) $$\varphi(\underline{x}, x_{n+1}) = \varphi(\underline{x}, 1) + m_{n+1}(x_{n+1}) - \varphi(\underline{1}, 1), \quad ((\underline{x}, x_{n+1}) \in P^{n+1}).$$

Namely if $\beta \in J(P^{n+1} \to A)$, then

$$2\beta\left(\underline{x}, \frac{t_1+t_2}{2}\right) = \beta(\underline{x}, t_1) + \beta(\underline{x}, t_2) \quad (\underline{x} \in P^n;\ t_1, t_2 \in P).$$

Let us substitute here in order $t_1 = 2x_{n+1}$, $t_2 = 2$;

$$t_1 = x_{n+1}+1, \quad t_2 = 1; \quad t_1 = x_{n+1}, \quad t_2 = 2,$$

then we obtain the equations

$$2\beta(\underline{x}, x_{n+1}+1) = \beta(\underline{x}, 2x_{n+1}) + \beta(\underline{x}, 2),$$

$$2\beta\left(\underline{x}, \frac{x_{n+1}+2}{2}\right) = \beta(\underline{x}, x_{n+1}+1) + \beta(\underline{x}, 1),$$

$$2\beta\left(\underline{x}, \frac{x_{n+1}+2}{2}\right) = \beta(\underline{x}, x_{n+1}) + \beta(\underline{x}, 2)$$

for all $\underline{x} \in P^n$, $x_{n+1} \in P$. Comparing these equations we obtain that β satisfies the functional equation

(11) $$\beta(\underline{x}, 2x_{n+1}) - 2\beta(\underline{x}, x_{n+1}) = \beta(\underline{x}, 2) - 2\beta(\underline{x}, 1), \quad (\underline{x} \in P^n,\ x_{n+1} \in P).$$

By $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F \in J(P^n \to A)$ it follows from (11) that

(12) $$\begin{cases} \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2x_{n+1}) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, x_{n+1}) = \\ = \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 1) = \\ = F(\underline{\lambda x}, 2\lambda_{n+1}) - F(\underline{x}, 2) - 2F(\underline{\lambda x}, \lambda_{n+1}) + 2F(\underline{x}, 1) \end{cases}$$

is valid for all $(\underline{x}, x_{n+1})$, $(\underline{\lambda}, \lambda_{n+1}) \in P^{n+1}$.

On the other hand from the definition of the function $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F$ we get

(13) $$\begin{cases} \Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, 2x_{n+1}) - 2\Delta_{(\underline{\lambda}, \lambda_{n+1})} F(\underline{x}, \lambda_{n+1}) = \\ = F(\underline{\lambda x}, 2\lambda_{n+1}x_{n+1}) - F(\underline{x}, 2x_{n+1}) - 2F(\underline{\lambda x}, \lambda_{n+1}x_{n+1}) + 2F(\underline{x}, x_{n+1}). \end{cases}$$

Equations (12) and (13) immediately imply that the function $\varphi\colon P^{n+1} \to A$ defined by (9) satisfies the functional equation

(14) $$\varphi(\underline{\lambda x}, \lambda_{n+1}x_{n+1}) - \varphi(\underline{x}, x_{n+1}) = \varphi(\underline{\lambda x}, \lambda_{n+1}) - \varphi(\underline{x}, 1)$$

for all $(\underline{x}, x_{n+1})$, $(\underline{\lambda}, \lambda_{n+1}) \in P^{n+1}$. From (14) by substitution $\lambda_{n+1} = 1$, $\underline{x} = \underline{1}$ we have

(15) $$\varphi(\underline{\lambda}, x_{n+1}) = \varphi(\underline{\lambda}, 1) + \varphi(\underline{1}, x_{n+1}) - \varphi(\underline{1}, 1) \quad (\underline{\lambda} \in P^n,\ x_{n+1} \in P).$$

Substituting (15) in (14) we easily obtain that

$$\varphi(\underline{1}, \lambda_{n+1}x_{n+1}) = \varphi(\underline{1}, \lambda_{n+1}) + \varphi(\underline{1}, x_{n+1})$$

for all $\lambda_{n+1}, x_{n+1} \in P$, i.e. the function $m_{n+1}: P \to A$ defined by

$$m_{n+1}(x_{n+1}) = \varphi(\underline{1}, x_{n+1}) \quad (x_{n+1} \in P)$$

is a homomorphism.

Thus (15) shows that φ is of form (10).

(ii) *The definition of function* φ *implies that*

$$\varphi(\underline{\lambda}\underline{x}, 1) - \varphi(\underline{x}, 1) = \Delta_{(\underline{\lambda}, 1)} F(\underline{x}, 2) - 2\Delta_{(\underline{\lambda}, 1)} F(\underline{x}, 1),$$

i.e. $\varphi(\underline{\lambda}\underline{x}, 1) - \varphi(\underline{x}, 1) \in J(P^n \to A)$ for all $\underline{\lambda} \in P^n$. By our assumption b) there is a function $\bar{\alpha} \in J(P^n \to A)$ and there are homomorphisms $\overline{m}_i: P \to A$ $(i=1, 2, \ldots, n)$, such that

$$\varphi(\underline{x}, 1) = \bar{\alpha}(\underline{x}) + \sum_{i=1}^{n} \overline{m}_i(x_i) \quad (\underline{x} \in P^n). \tag{16}$$

(iii) $\Delta_{(\underline{\lambda}, \lambda_{n+1})} F \in J(P^{n+1} \to A)$ implies that the function $\alpha_1: P^{n+1} \to A$ defined by

$$\alpha_1(\underline{x}, x_{n+1}) = F(\underline{x}, 2x_{n+1}) - F(\underline{x}, x_{n+1}) \quad \big((\underline{x}, x_{n+1}) \in P^{n+1}\big) \tag{17}$$

belongs to $J(P^{n+1} \to A)$.

On the other hand from (17) and (10) it follows also

$$\alpha_1(\underline{x}, x_{n+1}) = \varphi(\underline{x}, x_{n+1}) + F(\underline{x}, x_{n+1}) \quad \big((\underline{x}, x_{n+1}) \in P^{n+1}\big). \tag{18}$$

(iv) *Using* (18), (10) *and* (16) *we obtain* that

$$F(\underline{x}, x_{n+1}) = \alpha_1(\underline{x}, x_{n+1}) - \bar{\alpha}(\underline{x}) - \sum_{i=1}^{n+1} \overline{m}_i(x_i) - \varphi(\underline{1}, 1)$$

for all $(\underline{x}, x_{n+1}) \in P^{n+1}$.

Let α be the function defined by

$$\alpha(\underline{x}, x_{n+1}) = \alpha_1(\underline{x}, x_{n+1}) - \bar{\alpha}(\underline{x}) - \varphi(\underline{1}, 1) \quad \big((\underline{x}, x_{n+1}) \in P^{n+1}\big),$$

then $\alpha \in J(P^{n+1} \to A)$ and by the notation $m_i(x_i) = -\overline{m}_i(x_i)$ we obtain that

$$F(\underline{x}, x_{n+1}) = \alpha(\underline{x}, x_{n+1}) + \sum_{i=1}^{n+1} m_i(x_i) \quad \big((\underline{x}, x_{n+1}) \in P^{n+1}\big),$$

which corresponds to the form (8).

This completes the proof of theorem 1.1.

2. The general form of generalized Jensen-functions

The formula (7) given in the theorem 1.1. set the problem of the representation of generalized Jensen-functions. We shall demonstrate that if $P^n = \mathbf{R}_+^n$ (where $\mathbf{R}_+$ is the set of positive real numbers) and $A = \mathbf{R}$, then there is a close connection between the Jensen-functions of $J(\mathbf{R}_+^n \to \mathbf{R})$ and multiadditive functions. In this case we find the general form of Jensen-functions by the help of multiadditive functions.

To prove our theorem we need the following result of [5]:

Lemma. *If the function* $\alpha: \mathbf{R}_+ \to \mathbf{R}$ *belongs to* $J(\mathbf{R}_+ \to \mathbf{R})$, *then* α *has the form*

$$\alpha(x) = A(x) + b \quad (x \in \mathbf{R}_+), \tag{19}$$

where the function $A: \mathbf{R} \to \mathbf{R}$ *satisfies the Cauchy functional equation*

$$A(x+y) = A(x) + A(y) \quad ((x, y) \in \mathbf{R}^2) \tag{20}$$

(i.e. additive) and $b \in \mathbf{R}$ *is an arbitrary constant.*

Using induction on n, the number of variables, we shall prove

Theorem 2.1. *If the function* $\alpha: \mathbf{R}_+^n \to \mathbf{R}$ *belongs to* $J(\mathbf{R}_+^n \to \mathbf{R})$, *then* α *has the form*

$$\begin{cases} \alpha(\underline{x}) = A_n^0(x_1, \dots, x_n) + A_{n-1}^1(x_1, \dots, x_{n-1}) + \dots + A_{n-1}^n(x_2, \dots, x_n) + \\ + A_{n-2}^1(x_1, \dots, x_{n-2}) + \dots + A_{n-2}^{\frac{n(n-1)}{2}}(x_3, \dots, x_n) + \dots + A_2^1(x_1, x_2) + \\ + \dots + A_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n) + A_1^1(x_1) + \dots + A_1^n(x_n) + A_0 \end{cases} \tag{21}$$

for all $\underline{x} \in \mathbf{R}_+^n$ *where the functions* $A_i^k: \mathbf{R}^i \to \mathbf{R}$ $\left(i=1, \dots, n;\ k=0, \dots, \binom{n}{l};\ l=1, \dots, n-1\right)$ *are additive in each variable and* $A_0 \in \mathbf{R}$ *is an arbitrary constant.*

PROOF. a) *If* $n=1$, then the statement is true by the lemma.

b) Let us suppose that our statement holds for functions with n variables, then we prove that it is satisfied for $n+1$ variables too.

If α belongs to $J(\mathbf{R}_+^{n+1} \to \mathbf{R})$, then using the Jensen-property of α in the $(n+1)$th variable (because of our lemma) α has the form

$$\alpha(\underline{x}, x_{n+1}) = \bar{A}(\underline{x}, x_{n+1}) + b(\underline{x}) \tag{22}$$

for all $\underline{x} \in \mathbf{R}_+^n$, $x_{n+1} \in \mathbf{R}_+$, where the function $\bar{A}: \mathbf{R}_+^n \times \mathbf{R} \to \mathbf{R}$ is additive in the $(n+1)$th variable and $b: \mathbf{R}_+^n \to \mathbf{R}$ is an arbitrary function. Since α is a Jensen-function in each variable, using (22) we get the equation

$$\begin{cases} 2\bar{A}\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right), x_{n+1}\right) + 2b\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right)\right) = \bar{A}(\underline{x}_i(t_1), x_{n+1}) + \\ + \bar{A}(\underline{x}_i(t_2), x_{n+1}) + b(\underline{x}_i(t_1)) + b(\underline{x}_i(t_2)) \end{cases} \tag{23}$$

for all $\underline{x} \in \mathbf{R}_+^n$, $x_{n+1} \in \mathbf{R}$; $t_1, t_2 \in \mathbf{R}_+$ and $i=1, \dots, n$.

The additivity of $\bar{A}$ in the $(n+1)$th variable and (23) imply that

$$x_{n+1}\left[2\bar{A}\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right), 1\right) - \bar{A}(\underline{x}_i(t_1), 1) - \bar{A}(\underline{x}_i(t_2), 1)\right] = \\ = b(\underline{x}_i(t_1)) + b(\underline{x}_i(t_2)) - 2b\left(\underline{x}_i\left(\frac{t_1+t_2}{2}\right)\right)$$

is valid for all positive rational x_{n+1} and $i=1, \dots, n$; $\underline{x} \in \mathbf{R}_+^n$; $t_1, t_2 \in \mathbf{R}_+$. This is possibile if and only if the coefficient of x_{n+1} and the right hand side of this equation is equal to zero for all $\underline{x}\in\mathbf{R}^n_+$, $t_1, t_2\in\mathbf{R}_+$ and $i=1, \dots, n$, which shows that the function b belongs to $J(\mathbf{R}^n_+\to\mathbf{R})$.

By induction hypothesis it follows then that b has the form

$$(24)\qquad \begin{cases} b(\underline{x}) = B^0_n(\underline{x})+B^1_{n-1}(x_1, \dots, x_{n-1})+\dots+B^n_{n-1}(x_2, \dots, x_n)+ \\ +B^1_{n-2}(x_1, \dots, x_{n-2})+\dots+B_{n-2}^{\frac{n(n-1)}{2}}(x_3, \dots, x_n)+\dots+B^1_2(x_1, x_2)+ \\ +\dots+B_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n)+B^1_1(x_1)+\dots+B^n_1(x_n)+B_0 \end{cases}$$

for all $\underline{x}\in\mathbf{R}^n_+$, where the functions $B^k_i\colon \mathbf{R}^i\to\mathbf{R}$ are additive in each variable and $B_0\in\mathbf{R}$ is an arbitrary constant.

On the other hand $b\in J(\mathbf{R}^n_+\to\mathbf{R})$ and (23) imply that the function $\bar{A}$ has the Jensen-property in each of its first n variables, which means that for every fixed $x_{n+1}\in\mathbf{R}_+$ moreover for $x_{n+1}\in\mathbf{R}$ A belongs to $J(\mathbf{R}^n_+\to\mathbf{R})$. Thus by the induction hypothesis we have

$$(25)\qquad \begin{cases} \bar{A}(\underline{x}, x_{n+1}) = \tilde{A}^0_n(\underline{x}, x_{n+1})+\tilde{A}^1_{n-1}(x_1, \dots, x_{n-1}, x_{n+1})+\dots+ \\ +\tilde{A}^n_{n-1}(x_2, \dots, x_n, x_{n+1})+\dots+\tilde{A}^1_1(x_1, x_{n+1})+\dots+ \\ +\tilde{A}^n_1(x_n, x_{n+1})+\tilde{A}_0(x_{n+1}) \end{cases}$$

for all $\underline{x}\in\mathbf{R}^n_+$, $x_{n+1}\in\mathbf{R}$ where the functions $\tilde{A}^k_i\colon \mathbf{R}^i\to\mathbf{R}$ are additive in each variable $i\leqq n$, and $\tilde{A}_0\colon \mathbf{R}\to\mathbf{R}$ is an arbitrary function.

To complete the proof it is sufficient to show that every function $\tilde{A}^k_i$ is additive in the last variable too.

The function $\bar{A}$ is additive in the last variable, therefore from (25) we obtain the functional equation

$$(26)\qquad \begin{cases} \tilde{A}^0_n(x_1, \dots, x_n, t_1+t_2)+\tilde{A}^1_{n-1}(x_1, \dots, x_{n-1}, t_1+t_2)+\dots+ \\ +\tilde{A}^n_{n-1}(x_2, \dots, x_n, t_1+t_2)+\dots+\tilde{A}^1_1(x_1, t_1+t_2)+\dots+\tilde{A}^n_1(x_n, t_1+t_2)+ \\ +\tilde{A}_0(t_1+t_2) = \tilde{A}^0_n(x_1, \dots, x_n, t_1)+\tilde{A}^0_n(x_1, \dots, x_n, t_2)+ \\ +\tilde{A}^1_{n-1}(x_1, \dots, x_{n-1}, t_1)+\tilde{A}^1_{n-1}(x_1, \dots, x_{n-1}, t_2)+\dots+ \\ +\tilde{A}^n_{n-1}(x_2, \dots, x_n, t_1)+\tilde{A}^n_{n-1}(x_2, \dots, x_n, t_2)+\dots+\tilde{A}^1_1(x_1, t_1)+ \\ +\tilde{A}^1_1(x_1, t_2)+\dots+\tilde{A}^n_1(x_n, t_1)+\tilde{A}^n_1(x_n, t_2)+\tilde{A}_0(t_1)+\tilde{A}_0(t_2) \end{cases}$$

for every fixed $x_i\in\mathbf{R}_+$ $(i=1, \dots, n)$ and for all $t_1, t_2\in\mathbf{R}$. Using the multiadditivity we obtain from (26) that

$$\begin{gathered} x_1\cdot\ldots\cdot x_n[\tilde{A}^0_n(\underline{1}, t_1+t_2)-\tilde{A}^0_n(\underline{1}, t_1)-\tilde{A}^0_n(\underline{1}, t_2)]+ \\ +x_1\cdot\ldots\cdot x_{n-1}[\tilde{A}^1_{n-1}(\underline{1}, t_1+t_2)-\tilde{A}^1_{n-1}(\underline{1}, t_1)-\tilde{A}^1_{n-1}(\underline{1}, t_2)]+ \\ +\dots+x_2\cdot\ldots\cdot x_n\big[\tilde{A}^n_{n-1}(\underline{1}, t_1+t_2)-\tilde{A}^n_{n-1}(\underline{1}, t_1)-\tilde{A}^n_{n-1}(\underline{1}, t_2)\big]+ \\ +\dots+x_1[\tilde{A}^1_1(1, t_1+t_2)-\tilde{A}^1_1(1, t_1)-\tilde{A}^1_1(1, t_2)]+ \\ +\dots+x_n[\tilde{A}^n_1(1, t_1+t_2)-\tilde{A}^n_1(1, t_1)-\tilde{A}^n_1(1, t_2)] = \tilde{A}_0(t_1)+\tilde{A}_0(t_2)-\tilde{A}_0(t_1+t_2) \end{gathered}$$

for arbitrary positive rational numbers $x_1, \dots, x_n$ and for all $t_1, t_2 \in \mathbf{R}$. This is possibile if and only if the coefficients in the brachets and the right hand side of this equation is equal to zero for all $t_1, t_2 \in \mathbf{R}$. This means that the function $\tilde{A}_0: \mathbf{R} \to \mathbf{R}$ is additive on the whole plane.

If all $x_j's$ $(j=1, \dots, n)$ are rational except the ith $(i=1, \dots, n)$ one then using a similar reasoning then before we get from (26) that

$$x_1 \cdot \ldots \cdot x_{i-1} \cdot x_{i+1} \cdot \ldots \cdot x_n [\tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, 1, t_1+t_2) - \tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, 1, t_1) -$$

$$-\tilde{A}_n^0(1, \dots, 1, x_i, 1, \dots, t_2)] + \dots + x_1 \dots x_{i-1} x_{i+1} \dots x_{n-1} [\quad] +$$

$$+ \dots + x_2 \cdot \ldots \cdot x_{i-1} \cdot x_{i+1} \cdot \ldots \cdot x_n \big[\tilde{A}_{n-1}^n (1, \dots, x_i, \dots, 1, t_1+t_2) -$$

$$-\tilde{A}_{n-1}^n (1, \dots, x_i, \dots, 1, t_1) - \tilde{A}_{n-1}^n (1, \dots, x_i, \dots, 1, t_2)\big] + \dots +$$

$$+ x_1 [\tilde{A}_1^1(1, t_1+t_2) - \tilde{A}_1^1(1, t_1) - \tilde{A}_1^1(1, t_2)] + \dots + [\tilde{A}_1^i(x_i, t_1+t_2) - \tilde{A}_1^i(x_i, t_1) -$$

$$-\tilde{A}_1^i(x_i, t_2)] + \dots + x_n [\tilde{A}_1^n(1, t_1+t_2) - \tilde{A}_1^n(1, t_1) - \tilde{A}_1^n(1, t_2)] = 0$$

for $x_i \in \mathbf{R}_+$ $(i=1, \dots, n)$, $t_1, t_2 \in \mathbf{R}$. This implies that the functions $\tilde{A}_1^i: \mathbf{R}^2 \to \mathbf{R}$ $(i=1, \dots, n)$ are additive in the 2th variable for $x_i \in \mathbf{R}$ and by the extensions:

$$\tilde{A}_1^i(x_i, t) = -\tilde{A}_1^i(-x_i, t) \quad (x_i < 0); \quad \tilde{A}_1^i(0, t) = 0$$

this follows for all $x_i \in \mathbf{R}$ too and then the functions $\tilde{A}_1^i: \mathbf{R} \to \mathbf{R}$ are biadditive.

Let now $x_1, \dots, x_n$ be rational numbers except x_i and x_j, then from (26) we obtain that the functions $\tilde{A}_2^i: \mathbf{R}^3 \to \mathbf{R}$ $\left(i=1, \dots, \binom{n}{2}\right)$ are additive in each variable.

Continuing similarly we obtain that the functions $\tilde{A}_3^i: \mathbf{R}^4 \to \mathbf{R}, \dots, \tilde{A}_{n-1}^i: \mathbf{R}^n \to \mathbf{R}$ and finally $\tilde{A}_n^i: \mathbf{R}^{n+1} \to \mathbf{R}$ are additive in each variable, i.e. they are multiadditive functions.

From (22), (24) and (25) we obtain (21) for $n+1$ variables.

Thus the proof is complete.

3. The investigation of generalized difference-property in the case $P^n = \mathbf{R}_+^n$, $A = \mathbf{R}$

Knowing the general form of generalized Jensen-functions in the case $P^n = \mathbf{R}_+^n$ and $A = \mathbf{R}$, the theorem 1.1 can be formulated as follows:

Theorem 3.1. *Let* $F: \mathbf{R}_+^n \to \mathbf{R}$ *be a function, such that the function* $\Delta_{\underline{\lambda}} F: \mathbf{R}_+^n \to \mathbf{R}$ *defined by* (4) *belongs to* $J(\mathbf{R}_+^n \to \mathbf{R})$ *for all* $\underline{\lambda} \in \mathbf{R}_+^n$. *Then there exist homomorphisms* $m_i: \mathbf{R}_+ \to \mathbf{R}$ $(i=1, \dots, n)$, *multiadditive functions* $A_i^k: \mathbf{R}^i \to \mathbf{R}$ $(i=1, \dots, n;\ k=$

$=0, \ldots, \binom{n}{l};\ l=1, \ldots, n-1\Big)$ *and* $A_0 \in \mathbf{R}$ *constant, such that*

$$(27)\qquad \begin{cases} F(x_1, \ldots, x_n) = A_n^0(x_1, \ldots, x_n) + A_{n-1}^1(x_1, \ldots, x_{n-1}) + \ldots + \\ + A_{n-1}^n(x_2, \ldots, x_n) + \ldots + A_2^1(x_1, x_2) + \ldots + A_2^{\frac{n(n-1)}{2}}(x_{n-1}, x_n) + \\ + A_1^1(x_1) + \ldots + A_1^n(x_n) + \sum_{i=1}^{n} m_i(x_i) + A_0 \end{cases}$$

for all $\underline{x} \in \mathbf{R}_+^n$.

Proof. The statement of this theorem immediately follows from theorems 1.1. and 2.1. i.e. from the formulae (7) and (21).

In the case $n=2$ theorem 3.1. can be applied to determine the general solution of "rectangle-type" functional equations (see [6]), therefore it is interesting to reformulate our theorem in this case:

Corollarly. Let $F: \mathbf{R}_+^2 \to \mathbf{R}$ be a function, such that the function $\Delta_{\lambda,\mu} F: \mathbf{R}_+^2 \to \mathbf{R}$ defined by $\Delta_{\lambda,\mu} F(x, y) = F(\lambda x, \mu y) - F(x, y)$ $(x, y, \lambda, \mu \in \mathbf{R}_+)$ belongs to $J(\mathbf{R}_+^2 \to \mathbf{R})$ for all $(\lambda, \mu) \in \mathbf{R}_+^2$. Then there exist homomorphisms $m_1, m_2: \mathbf{R}_+ \to \mathbf{R}$, a biadditive function $A: \mathbf{R}^2 \to \mathbf{R}$, additive functions $a_1, a_2: \mathbf{R} \to \mathbf{R}$ and a constant $b \in \mathbf{R}$, such that

$$F(x, y) = A(x, y) + a_1(x) + a_2(y) + m_1(x) + m_2(y) + b \quad (x, y \in \mathbf{R}_+).$$

References

[1] J. Aczél, Lectures on Functional Equations and their Applications, *New York—London,* 1966.

[2] N. G. de Bruijn, Functions whose differences belong to a given class, *Nieuw Arch. Wisk.*, (2) **23**, (1951) 194—218.

[3] F. W. Carroll, Difference properties for continuity and Riemann integrability on locally compact groups, *Trans. Am. Math. Soc.* **102**, (1962) 284—292.

[4] F. W. Carroll—F. S. Koehl, Difference properties for Banach-valued Functions on compact Groups, *Indagationes Math.*, **31**, (1969) 327—332.

[5] Z. Daróczy—K. Lajkó—L. Székelyhidi, Functional Equations on ordered Fields, *Publ. Math. (Debrecen)* **24**. (1977).

[6] O. Em. Gheorghiu, Über einige Rechteckförmige Funktionalgleichungen, *Bul. Mat. de la Soc. Sci. Math. de la R. S. de Roumanie*, **16** (64), (1972) 429—436.

(Received April 23, 1975.)