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On Riemannian manifolds endowed with a $\mathcal{T}$-parallel almost contact 4-structure

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Abstract. $\mathcal{T}$-parallel almost contact 4-structures on a Riemannian manifold are studied. It is proved that such a manifold is a local Riemannian product of two totally geodesic submanifolds, one of them being a space form. Additional results are obtained when the manifold is endowed with a framed f-structure.

1. Introduction

In the last two decades, contact, almost contact, paracontact and almost cosymplectic manifolds carrying r $(r > 1)$ Reeb vector fields ξ_r have been studied by a certain number of authors, as for instance: M. Kobayashi [11], A. Bucki [4], S. Tachibana and W. N. Yu [22], K. Yano and M. Kon [25], V. V. Goldberg and R. Rosca [8] and some others.

In the present paper we consider a $(2m+4)$ dimensional Riemannian manifold carrying 4 structure vector fields ξ_r $(r, s \in \{2m+1, \dots, 2m+4\})$ and with a distinguished vector field $\mathcal{T}$, such that the vertical connection forms define a $\mathcal{T}$-parallel connection and the Reeb vector fields are $\mathcal{T}$-parallel (this structure is called a $\mathcal{T}$-parallel almost contact 4-structure and it will be defined in Definition 3.1). Then we shall prove that such a manifold is a local Riemannian product of two totally geodesic submanifolds, $M = M^{\top} \times M^{\perp}$, where $M^{\perp}$ is a space form tangent to the distribution generated by the Reeb vector fields, and that the vector field $\mathcal{T}$ is closed torse forming (Theorem 3.3).

In section 4 we shall study conformal-type structures induced by a $\mathcal{T}$-parallel almost contact 4-structure. Finally, in section 5 we assume that the manifold under consideration is endowed with a framed f-structure, proving that $M^{\top}$ is a Kählerian submanifold (Theorem 5.2).

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2. Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator defined by the metric tensor g. We assume that M is oriented and ∇ is the Levi-Civita connection. Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \to T^*M$, $X \to X^\flat$, the *musical isomorphism* defined by g. Next, following a standard notation, we set: $A^q(M, TM) = \mathrm{Hom}(\Lambda^q TM, TM)$ and notice that elements of $A^q(M, TM)$ are vector valued q-forms ($q \leq \dim M$). Denote by d^∇ : $A^q(M, TM) \to A^{q+1}(M, TM)$ the *exterior covariant derivative* operator with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$). The identity tensor field I of type (1,1) can be considered as a vector valued 1-form $I \in A^1(M, TM)$ (and it is also called the *soldering form* [7]).

We shall remember the following

Definition 2.1. (1) (see [10]) The operator $d^\omega = d + e(\omega)$ acting on ΛM is called the *cohomology operator*, where $e(\omega)$ means the exterior product by the closed 1-form $\omega \in \Lambda^1 M$, i.e., $d^\omega u = du + \omega \wedge u$ for any $u \in \Lambda M$. One has $d^\omega \circ d^\omega = 0$, and if $d^\omega u = 0$, u is said to be d^ω-*closed*. If ω is exact, then u is said to be d^ω-*exact*.

(2) (see [18], [16]) Any vector field $X \in \Gamma TM$ such that: $d^\nabla(\nabla X) = \nabla^2 X = \pi \wedge I \in A^2(M; TM)$ for some 1-form π, is called an *exterior concurrent vector field* and the 1-form π, which is called the *concurrence form*, given by $\pi = f X^\flat$, $f \in C^\infty(M)$.

(3) (see [23], [16]) A vector field $\mathcal{T}$ whose covariant differential satisfies $\nabla \mathcal{T} = rI + \alpha \otimes \mathcal{T}$; $r \in C^\infty(M)$ where $\omega = \mathcal{T}^\flat$ is a closed form, is called a *closed torse forming*.

If $\Re$ denotes the Ricci tensor of ∇ and X an exterior concurrent vector field, one has $\Re(X, Z) = -(n-1) f g(X, Z)$, $Z \in \Gamma TM$, $n = \dim M$.

Let C be any conformal vector field on M (i.e., the conformal version of Killing's equations). As is well known, C satisfies

$$\mathcal{L}_C g(C, Z) = \rho g(C, Z) \text{ or } g(\nabla_Z C, Z') + g(\nabla_Z C, Z) = \rho g(Z, Z') \tag{2.1}$$

$(Z, Z' \in \Gamma TM)$ where the conformal scalar ρ is defined by $\rho = \frac{2}{n}(\operatorname{div} C)$.

We recall the following basic formulas (see [3])

Proposition 2.2. *With the above notation, let $\mathcal{L}_C$, K, Δ and $\Re$ denote the Lie derivative with respect to C, the scalar curvature, the Laplacian and the Ricci tensor field of ∇, respectively. Then:*
(1) $\mathcal{L}_C Z^\flat = \rho Z^\flat + [C, Z]^\flat$ *(Orsted's lemma).*
(2) $\mathcal{L}_C K = (n-1)\Delta\rho - K\rho$.
(3) $2\mathcal{L}_C \Re(Z, Z') = \Delta\rho g(Z, Z') - (n-2)(\mathrm{Hess}_\nabla \rho)(Z, Z')$ *where* $(\mathrm{Hess}_\nabla \rho)(Z, Z') = g(Z, \nabla_{Z'}(\mathrm{grad}\,\rho))$.

Definition 2.3 (see [19], [20], [15]). Any vector field C whose covariant differential satisfies $\nabla C = fI + C \wedge X$ is said to be a *skew-symmetric conformal* (ab. SKC) vector field or a *structure conformal* vector field, where $\wedge$ means the wedge product of vector fields, i.e., $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y;\ X, Y, Z \in \Gamma(TM)$.

Remark 2.4. Let $\mathcal{O} = \mathrm{vect}\{e_A; A \in 1, \dots, n\}$ be an adapted local field of orthonormal frames on M and let $\mathcal{O}^* = \mathrm{covect}\{\omega^A\}$be the associated coframe. With respect to $\mathcal{O}$ and $\mathcal{O}^*$ the soldering form I and E. Cartan's structure equations can be written in indexless manner as

(1) $I = \omega^A \otimes e_A \in A^1(M, TM)$
(2) $\nabla e = \vartheta \otimes e \in A^1(M, TM)$
(3) $d\omega = -\vartheta \wedge \omega$
(4) $d\vartheta = -\vartheta \wedge \vartheta + \Theta$

In the above equations ϑ (resp. Θ) are the local connection forms in the bundle $O(M)$ (resp. the curvature form on M).

Finally, we remember the following

Proposition 2.5. *Let $\pi \in \Lambda^1 M$ be a Pffaf form on a manifold M. Then in order that π be of class $2s$ on M it is necessary and sufficient to have* $(d\pi)^{s+1} = 0,\ \pi \wedge (d\pi)^s = 0$.

3. The main result

Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional oriented Riemannian manifold carrying 4 Reeb vector fields ξ_r $(r, s \in \{2m+1, \dots, 2m+4\})$ with associated structure covectors η^r, that is $\eta^r(\xi_s) = \delta_{rs}$. Following a known terminology we may decompose the tangent space $T_p(M)$ at $p \in M$ to M as $T_pM = D_p^\top \oplus D_p^\perp$.Then $D_p^\perp$ is a 4-dimensional distribution defined by the set $\{\xi_r\}$, called the *vertical distribution*, and its orthogonal complement $D_p^\top = \{\xi_r\}^\perp$ which is called the *horizontal distribution*. Consequently any vector field $Z \in \Gamma(TM)$ may be written as $Z = (Z - \eta^r(Z)\xi_r) + \eta^r(Z)\xi_r = Z^\top + Z^\perp$ where $Z^\top$ (resp. $Z^\perp$) is the horizontal component of Z (resp. the vertical component of Z). We recall that setting $A; B \in \{1, 2, \dots, 2m\}$ the connection forms ϑ^A_B, ϑ^r_B and ϑ^r_s are called the horizontal, the transversal and the vertical connection forms respectively (see also [21]).

With the above notation, one has the following

Definition 3.1 ([17], [9]). Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional oriented Riemannian manifold carrying 4 Reeb vector fields ξ_r such that the vertical connection forms verifies $\vartheta^r_s = \langle \mathcal{T}, \xi_s \wedge \xi_r \rangle$, where $\mathcal{T}$ is a certain vertical vector field. Then, we say that vertical connection forms ϑ^r_s define on $D^\perp$ a *$\mathcal{T}$-parallel connection* and $\mathcal{T}$ is called the *generator* of the considered $(\mathcal{T}.P)$-connection. Moreover, if the Reeb vector fields are $\mathcal{T}$-parallel, i.e., $\nabla_{\mathcal{T}} \xi_r = 0$, then the manifold $M(\xi_r, \eta^r, g)$ is said to be endowed with a *$\mathcal{T}$-parallel almost contact 4-structure* (abr. $\mathcal{T}$.P.A.C. 4-structure).

In the present paper we shall deal with these manifolds.

Remark 3.2. If we set $\mathcal{T} = \sum t_r \xi_r;\ t_r \in C^\infty(M)$ then the vertical connection forms are expressed by $\vartheta^r_s = t_s \eta^r - t_r \eta^s$. Since the vertical connection forms satisfy $\vartheta^r_s(\mathcal{T}) = 0$, then by reference to [13] we may say that ϑ^r_s are *relations of integral invariance* for the vector field $\mathcal{T}$.

Similarly one may decompose in an unique fashion the soldering form I of M as $I = I^\top + I^\perp$ where $I^\top = \omega^A \otimes e_A$ and $I^\perp = \eta^r \otimes \xi_r$ mean the line element of $D^\top$ and the line element of $D^\perp$ respectively.

We can state

Theorem 3.3. *Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional Riemannian manifold endowed with a $\mathcal{T}$-parallel almost contact 4-structure and let $\mathcal{T}$ be the generator vector field of this structure.*

For such a manifold the structure covectors η^r ($r \in \{2m+1, \dots, 2m+4\}$) are of class 2 and cohomologically exact, i.e., $d^{-\omega}\eta^r = 0$, where ω is the dual form of the generator $\mathcal{T}$ which enjoys the property to be a closed torse forming and to define a relative infinitesimal conformal transformation of the almost contact structure of M.

Any manifold M which carries a ($\mathcal{T}$.P.A.C.) 4-structure may be viewed as the local Riemannian product $M = M^\top \times M^\perp$ such that:

(i) $M^\perp$ is a totally geodesic submanifold of M, tangent to the vertical distribution $D^\perp = \{\xi_r\}$ which enjoys the property to be a space form of curvature $-2a$ ($a =$ const.)

(ii) $M^\top$ is a totally geodesic submanifold of M, tangent to the horizontal distribution $D^\top = \{\xi_r\}^\perp$ of M.

Proof. Making use of the structure equations of Remark 2.4(2) and taking account of Remark 3.2 one derives:

$$\nabla \xi_r = t_r I^\perp - \eta^r \otimes \mathcal{T}. \tag{3.1}$$

Hence if $Z_1^\perp, Z_2^\perp \in D_p^\perp$ are any vertical vector fields, it quickly follows from (3.1) $\nabla_{Z_2^\perp} Z_1^\perp \in D_p^\perp$. This, as is known, proves that $D_p^\perp$ is an *autoparallel* foliation and that the leaves $M^\perp$ of $D_p^\perp$ are totally geodesic submanifolds of M (in our case, $\dim M^\perp = 4$). Next making use of the structure equations of Remark 2.4(3) one finds

$$d\eta^r = \omega \wedge \eta^r \tag{3.2}$$

where $\omega = \mathcal{T}^\flat$ denotes the dual form of the generator vector field $\mathcal{T}$.

By reference to [7], equations (3.2) show that all the Reeb covectors η^r are *exterior recurrent* and by a simple argument it follows that the recurrence form ω is necessarly closed, i.e., $d\omega = 0$. With the help of (3.1) and (3.2) one also derives from $I^\perp = \eta^r \otimes \xi_r$ that $I^\perp$ is exterior covariant closed, i.e., $d^\nabla(I^\perp) = 0$ and this is matching the fact that $I^\perp$ is the soldering form of the leaf $M^\perp$. By reference to Proposition 2.5 it is seen by (3.2) that the structure covectors η^r are of class 2.

Let now denote by $\varphi = \eta^{2m+1} \wedge \ldots \wedge \eta^{2m+4}$ the simple form which coresponds to $D_p^\perp$ (or equivalently the volume element of $M^\perp$). By (3.2) one has at once $d\varphi = 0$ and therefore since one may write $D_p^\top \subset \ker(\varphi) \cap \ker(d\varphi)$ we conclude that the horizontal distribution $D_p^\top$ is also involutive. Then setting $M^\top$ for the $2m$ leaf of $D_p^\top$, it is seen that ξ_r are geodesic normal section for the immersion $\kappa : M^\top \to M$, which is totally geodesic. It follows from the above discussion that the manifold M under consideration is the local product $M = M^\top \times M^\perp$, where $M^\top$ and $M^\perp$ are totally geodesic submanifolds of M, tangent to the horizontal distribution $D^\top$ and the vertical distribution $D^\perp$ of M respectively.

Further since the dual form ω of $\mathcal{T}$ is expressed by $\omega = t_r \eta^r$ then by virtue of (3.2) one may set

$$dt_r = \lambda \eta^r \implies d\lambda - \lambda\omega = 0 \tag{3.3}$$

which shows that ω is an exact form. In consequence of this fact, equations (3.2) may be expressed, using the notation introduced in Definition 2.1(1), as $d^{-\omega}\eta^r = 0$, thus proving that the structure covectors of $M(\xi_r, \eta^r, g)$ are cohomologically exact.

Taking now the covariant differential of the generator vector field $\mathcal{T}$, one derives on behalf of (3.1) and (3.3)

$$\nabla\mathcal{T} = (\lambda + 2t)I^{\perp} - \nu \otimes \mathcal{T};\ 2t = \|\mathcal{T}\|^2 \tag{3.4}$$

which shows the significative fact that $\mathcal{T}$ is a closed torse forming (def. 2.1(3)). Since this quality implies that $\mathcal{T}$ is a gradient vector field, this fact is in accordance with equation (3.3). We also derive from (3.4)

$$dt = \lambda\omega \implies t + \lambda = a = \text{const.} \tag{3.5}$$

Next operating on (3.1) by the exterior covariant derivative operator d^{∇} one quickly derives by (3.2) and (3.4) that one has $d^{\nabla}(\nabla\xi_r) = \nabla^2\xi_r = 2a\eta^r \wedge I^{\perp}$. The above equations reveal the important fact that all the vectors $\{\xi_r\}$ on $M^{\perp}$ are exterior concurrent vector fields (see [20]). Then since the conformal scalar $2a$ is constant, we conclude by reference to [16] that the vertical submanifold $M^{\perp}$ is a *space form* of curvature $-2a$.

Next by (3.2), (3.3) and (3.5) one derives succesively $\mathcal{L}_{\mathcal{T}}\eta^r = (a + t)\eta^r - t_r\omega$ and $d(\mathcal{L}_{\mathcal{T}}\eta^r) = (2a + \lambda)\omega \wedge \eta^r$. In consequence of the last equation and by reference to [14] we agree to say that the generator vector $\mathcal{T}$ defines a relative infinitesimal conformal transformation of the considered almost contact 4-structure, thus finishing the proof.

4. Conformal-type structures induced by a ($\mathcal{T}$.P.A.C.) 4-structure

In the present section we consider on $M^{\perp}$ the 2-form ψ of rank 2 (if $\Omega \in \Lambda^2 M$, rank r is the smallest integer such that $\Omega^{r+1} = 0$), defined by $\psi = \eta^{2m+1} \wedge \eta^{2m+2} + \eta^{2m+3} \wedge \eta^{2m+4}$. On behalf of (3.2) one quickly derives by exterior differentiation of ψ that $d\psi = 2\omega \wedge \psi \Leftrightarrow d^{-2\omega}\psi = 0$ (the last equality obtained on behalf of Definition 2.1(1)). Therefore following a known definition it is seen that ψ is a conformal symplectic form on $M^{\perp}$ having ω (resp. $\mathcal{T}$) as covector of Lee (resp. vector field of Lee). In addition in the case under discussion one may say that ψ is a $d^{-2\omega}$-*exact form.*

It should be noticed that this property is in accordance with the general properties of $\mathcal{T}$-parallel connections (see also [14]). If $Y \in \Gamma TM^{\perp}$ is any vertical vector field, then by reference to [12] we set ${}^{b}Y = -i_Y\psi$. Do not confuse with the the musical isomorphism $\flat : \Gamma TM \to \Gamma TM^*$, which is denoted by $X \to X^{\flat}$. For instance, $\omega = \mathcal{T}^{\flat}$.

In the case under discussion and in order to simplify we write

$$\beta = -{}^b\mathcal{T} = t_{2m+1}\eta^{2m+2} + t_{2m+3}\eta^{2m+4} - t_{2m+2}\eta^{2m+1} - t_{2m+4}\eta^{2m+3}$$

and by (3.3) and (3.2) one gets $d\beta = 2\lambda\psi + \omega\wedge\beta$ by which after a standard calculation one derives $\mathcal{L}_{\mathcal{T}}\psi = 2(a+t)\psi - \omega\wedge\beta$. Since ω is an exact form, then following [1] the above equation shows that $\mathcal{T}$ defines a *weak infinitesimal conformal transformation* of ψ. Then we obtain $d(\mathcal{L}_{\mathcal{T}}\psi) = 8a\omega\wedge\psi$. Therefore we may also say that $\mathcal{T}$ defines a *relative infinitesimal conformal transformation* of ψ.

Consider now the vertical vector field $C = C^r\xi_r$ and set $\varrho = {}^bC$. Then in order that C be an infinitesimal conformal transformation of ψ, one finds making use of (3.2)

$$dC^r = C^r\omega. \tag{4.1}$$

This implies $d\varrho = 2\omega\wedge\varrho \Leftrightarrow d^{-2\omega}\varrho = 0$ and setting $s = g(C,\mathcal{T})$ one may write $\mathcal{L}_{\mathcal{T}}\psi = 2s\psi$. In the light of this problem, and making use of (3.1) and (4.1) one derives

$$\nabla C = sI^{\perp} + C\wedge\mathcal{T} \tag{4.2}$$

which reveals the important fact that C is a *structure conformal* vector field having $2s = \rho$ as conformal scalar (see Definition 2.3). Setting $\alpha = C^\flat$ one finds by (3.4) and (4.2)

$$ds = \lambda\alpha + s\omega \tag{4.3}$$

and on the other hand by (3.2) one has

$$d\alpha = 2\omega\wedge\alpha \iff d^{-2\omega}\alpha = 0. \tag{4.4}$$

Hence one may say that as ψ the dual form α of C is $d^{-2\omega}$-exact. It should be noticed that equation (4.4) is in accordance with the general properties of structure conformal vector fields [19] (see also [14], [15]).

By (3.3), (4.3) and (4.4) it is seen that the existence of the structure conformal vector field C is determined by the exterior differential system Σ_e whose *characterisitic numbers* are $r = 3$, $s_0 = 2$, $s_1 = 1$. Since $r = s_0 + s_1$ it follows by E. Cartan's test [5] that Σ_e is *involutive* and C is determined by 1 arbitrary function of 2 arguments.

Next since $\rho = 2s$, it follows at once from (4.3), by duality: $\operatorname{grad}\rho = 2\lambda C + \rho\mathcal{T}$. But as it is known $\operatorname{div} Z = tr[\nabla Z]$, $Z \in \Gamma TM$, and so one gets from (3.4) $\operatorname{div}\mathcal{T} = 4a + 2t$ and C being a conformal vector field one has

$\operatorname{div} C = 4\rho$. Therefore by the general formulas $\Delta f = -\operatorname{div}(\operatorname{grad} f)$, $f \in C^\infty M$, a short calculation gives

$$\Delta\rho = -8a\rho \tag{4.5}$$

which shows that ρ is an *eigenfunction* of Δ and has $-8a$ associated *eigen value.* Following a known theorem, it follows that if $M^\perp$ is compact, then necessarily $a = -\mu^2$ ($\mu =$ const.), that is, $M^\perp$ is an elliptic submanifold of M.

On the other hand taking the covariant differential of $\operatorname{grad}\rho$, then by a standard calculation one infers

$$\nabla \operatorname{grad}\rho = 4a\rho I^\perp \tag{4.6}$$

which reveals that $\operatorname{grad}\rho$ is *concurrent* vector field on $M^\perp$ [6] (we recall that concurrency is of conformal nature). Accordingly on behalf of the definition given in [14], we may say in the case under consideration C has the *divergence conformal property*. It is worth to point out that if $M^\perp$ is an elliptic submanifold of M (i.e., $a = -\mu^2$), then following Obata's theorem [24], $M^\perp$ is *isometric* to a sphere of radius $\frac{1}{2}\mu$.

Further since $M^\perp$ is a space form, then we recall [16] that any vector field on $M^\perp$ is E.C., with the same conformal scalar $2a$. Consequently, if $\Re$ denotes the Ricci tensor of ∇, one has

$$\Re(C, Z) = -6ag(C, Z),\ Z \in \Gamma TM^\perp. \tag{4.7}$$

Then by (4.5), (4.6), (4.7) and making use of Proposition 2.2(3) and carrying out the calculations one derives $\mathcal{L}_C g(C, Z) = \frac{4}{3}\rho g(C, Z)$. Therefore one may state that the (S.C)-vector field C defines an infinitesimal conformal transformation of all the functions $g(C, Z)$ where $Z \in \Gamma TM^\perp$. It should be noticed that this situation is similar to that of [14]. In addition by (3.1) and (4.2) one finds

$$[C, \xi_r] = -\frac{\rho}{2}\xi_r \tag{4.8}$$

which shows that the structure vector fields ξ_r admit *infinitesimal transformations* of generator C. Next making use of Orsted's lemma (Proposition 2.2(1)) it follows

$$\mathcal{L}_C \eta^r = \rho\eta^r. \tag{4.9}$$

Hence making use of a known terminology, it follows that C defines an *almost contact transformation* of the structure covectors η^r.

Finally we denote by $\mathcal{P} = \xi_{2m+1} \wedge \xi_{2m+2} + \xi_{2m+3} \wedge \xi_{2m+4}$ the *Poisson bivector* [12] associated with the conformal symplectic form ψ. Since $\mathcal{P}$ may be expressed as

$$\mathcal{P} = \eta^{2m+2} \otimes \xi_{2m+1} - \eta^{2m+1} \otimes \xi_{2m+2} \\ + \eta^{2m+4} \otimes \xi_{2m+3} - \eta^{2m+3} \otimes \xi_{2m+4}$$

then since the Lie derivative is additive, one gets by (4.8) and (4.9) that $\mathcal{L}_C \mathcal{P} = 0$ which shows that C defines an infinitesimal automorphism of $\mathcal{P}$.

Next operating on the vector valued 1-form $\mathcal{P}$ by the operator d^∇ one derives after two sucesive computations $d^\nabla \mathcal{P} = \omega \wedge \mathcal{P} - 2\psi \otimes \mathcal{T} - \beta \wedge I^\perp \in A^2(M, TM)$ ($\beta = -{}^b\mathcal{T}$) and $d^{\nabla^2} \mathcal{P} = 4a\psi \wedge I^\perp$. Therfore (see Proposition 2.5) the last equality shows that $\mathcal{P}$ is a *2-exterior vector valued 1-form*. Moreover, taking into account $\mathcal{L}_\mathcal{T} \psi = 2s\psi$ a short calculation gives $\mathcal{L}_C(d^{\nabla^2}\mathcal{P}) = \frac{\rho}{2} d^{\nabla^2}\mathcal{P}$ that is C defines an infinitesimal conformal transformation of $d^{\nabla^2}\mathcal{P}$.

Then one has the

Theorem 4.1. *Let $M(\xi_r, \eta^r, g)$ be a $(2m+4)$-dimensional Riemannian manifold endowed with a $(\mathcal{T}$.P.A.C.$)$ 4-structure discussed in Section 2 and having $\mathcal{T}$ as generator vector field. Let $M^\perp$ be the space form submanifold of M, tangent to the vertical distribution $D^\perp = \{\xi_r\}$ of M. One has the following properties:*

(i) $M^\perp$ is equipped with a conformal symplectic structure CSp(4,**R**) *defined by the form $\psi \in \Lambda^2 M^\perp$ (of* rank 2*) and such that the covector of Lee corresponding to* CSp(4,**R**) *is the dual form ω of $\mathcal{T}$, that is, $d\psi = 2\omega \wedge \psi$ and $\mathcal{T}$ defines a relative infinitesimal conformal transformation of ψ, that is, $d(\mathcal{L}_\mathcal{T}\psi) = 8a\omega \wedge \psi$, $(a = const.)$*

(ii) Any vector field C which defines an infinitesimal conformal transformation of ψ is a structure conformal vector field, i.e., $\nabla C = g(\mathcal{T}, C) I^\perp + C \wedge \mathcal{T}$ and one has $\mathcal{L}_C \psi = \rho\psi$; $\rho = 2g(\mathcal{T}, C)$ and $\mathcal{L}_C g(C, Z) = \frac{4}{3}\rho g(C, Z)$, $Z \in \Gamma T M^\perp$.

(iii) The conformal scalar ρ $(\mathcal{L}_C g = \rho g)$ is an eigenfunction of Δ and if $M^\perp$ is compact, then $a = -\mu^2$ and $M^\perp$ is isometric to a sphere of radius $\frac{1}{2}\mu$.

(iv) The Poisson bivector $\mathcal{P}$ associated with ψ is a 2-exterior vector valued 1-form, i.e., $d^{\nabla^2}\mathcal{P} = 4a\psi \wedge I^\perp$ and C defines an infinitesimal automorphism of $\mathcal{P}$.

5. Framed f-structures

In the present section we assume that the manifold $M(\xi_r, \eta^r, g)$ under consideration is endowed with a framed f-structure ϕ [27], that is ϕ is a tensor field of type (1,1) and rank $2m$ which satisfies:

(1) $\phi^3 + \phi = 0$

(2) $\phi^2 = -I + \sum \eta^r \otimes \xi_r$; $\phi\xi_r = 0$; $\eta^r \circ \phi = 0$

(3) $g(Z, Z') = g(\phi Z, \phi Z') + \sum \eta^r(Z)\eta^r(Z')$; $Z, Z' \in \Gamma TM$ and the fundamental 2-form Ω associated with the f-structure satisfies:

(4) $\Omega(Z, Z') = g(\phi Z, Z')$; $\Omega^m \wedge \varphi \neq 0$, φ being the volume element of $M^\perp$, i.e., $\varphi = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \wedge \eta^{2m+4}$.

Such a manifold $M(\phi, \Omega, \xi_r, \eta^r, g)$ is, as known, defined as *framed f-manifold*.

With respect to the cobasis $\mathcal{O}^* = \text{covect}\{\omega^A, \eta^r\}$ the form Ω is expressed by $\Omega = \sum \omega^a \wedge \omega^{a^*}$; $a \in \{1, \dots, m\}$; $a^* = a+m$ and the horizontal connection forms ϑ^A_B satisfies the known Kählerian conditions

$$\vartheta^a_b = \vartheta^{a^*}_{b^*};\ \vartheta^{a^*}_b = \vartheta^{b^*}_a. \tag{5.1}$$

Since on the other hand by (3.1) it is seen that the transversal connection forms ϑ^r_A vanish, one gets by exterior differentiation $d\Omega = 0$. Since Ω is of constant rank and closed it follows that it is a *presymplectic* form on M and a symplectic form on $M^\top$. We notice that in this case $\ker(\Omega)$ coincides with the vertical distribution $D^\perp_p$ of M which may be also called *characteristic* distribution of Ω. In addition by condition (3) of a framed f-structure and $\vartheta^r_A = 0$ one has $(\nabla\phi)Z = 0$, $Z \in \Gamma TM$, that is ∇ and ϕ commute.

Recall now that the *torsion tensor* field S of an f-structure is the vector valued 2-form defined by $S = N_\phi + S^\perp$ where $N_\phi(Z, Z') = [\phi Z, \phi Z'] + \phi^2[Z, Z'] - \phi[Z, \phi Z'] - \phi[\phi Z, Z']$ is the Nijenhuis tensor field, and $S^\perp = 2\sum d\eta^r \otimes \xi_r$ is the vertical component of S. By (3.10), (5.6) and $(\nabla\phi)Z = 0$ it is easily seen that S vanishes on $D^\top$. In this case, the f-structure (ϕ, ξ_r, η^r) is said to be *horizontal-normal* (or $D^\top$-normal) [2].

Consequently, following a definition of A. Bejancu [2] the framed f-manifold $M(\phi, \Omega, \xi_r, \eta^r, g)$ under consideration is a *framed-CR manifold*. On the other hand, taking into account that Ω is closed, the horizontal submanifold $M^\top$ of M moves to a symplectic submanifold.

It also should be noticed that by (3.2) one may write $S^\perp$ as $S^\perp = 2\omega \wedge I^\perp \Rightarrow d^\nabla S^\perp = 0$ that is, $S^\perp$ is a closed vector valued 2-form. We agree with the following

Definition 5.1. Let M be a framed f-manifold and let $S^{\perp}$ be the vertical component of its associated torsion tensor. If the covariant differential of $S^{\perp}$ is closed, i.e., $d^{\nabla} S^{\perp} = 0$, we say that M is a *vertical closed framed f-manifold*.

Now since one finds $\mathcal{L}_{\mathcal{T}} \xi_r = [\mathcal{T}, \xi_r] = t_r \mathcal{T} - (t+a)\xi_r$ then one get at once $\mathcal{L}_{\mathcal{T}} S^{\perp} = 2\lambda S^{\perp}$. Accordingly the Lee vector field $\mathcal{T}$ defines an infinitesimal conformal transformation of $S^{\perp}$.

Then we can state the following

Theorem 5.2. *Let $M(\phi, \Omega, \xi_r, \eta^r, g)$ be a framed f-manifold endowed with a $\mathcal{T}$-parallel almost contact 4-structure, and let $S^{\perp}$ be the vertical component of the torsion tensor field S associated with the f-structure defined by ϕ.*

*Any such M is a framed f-*CR *manifold which is vertical torsion closed, i.e., $d^{\nabla} S^{\perp} = 0$, and may be viewed as the local Riemannian product $M = M^{\top} \times M^{\perp}$ such that:*

(i) $M^{\top}$ is a totally geodesic Kählerian submanifold of M, tangent to $\{\xi_r\}^{\perp}$;

(ii) $M^{\perp}$ is a totally geodesic space form submanifold of M, tangent to $\{\xi_r\}$;

(iii) the Lee vector field $\mathcal{T}$ of the ($\mathcal{T}$.P.A.C.) *4-structure defines an infinitesimal conformal transformation of $S^{\perp}$.*

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