Non-additive Ring and Module Theory, III.
Morita Equivalences

Dedicated to the memory of Andor Kertész

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This paper is a continuation of [19], [20]. References are quoted there.

As in ring theory one may ask the question when two categories ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$ for monoids A and B are equivalent. Now in ring theory we know from the additivity of the equivalences $\mathscr{F}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G}: {}_B\mathscr{C} \to {}_A\mathscr{C}$ that the natural bijections

$$\mathrm{Hom}_B(\mathscr{F}(M), N) \cong \mathrm{Hom}_A(M, \mathscr{G}(N)) \quad \text{and} \quad \mathrm{Hom}_A(\mathscr{G}(N), M) \cong \mathrm{Hom}_B(N, \mathscr{F}(M))$$

are isomorphisms of abelian groups. So in the general case we only want to consider equivalences such that there are isomorphisms ${}_B[\mathscr{F}(M), N] \cong {}_A[M, \mathscr{G}(N)]$ and ${}_A[\mathscr{G}(N), M] \cong {}_B[N, \mathscr{F}(M)]$. In view of theorem 4.3. this is equivalent to studying equivalences such that $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{C}$-functors. The last condition can be studied even in monoidal, non-closed categories.

We call ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$ *$\mathscr{C}$-equivalent* if there are inverse equivalences $\mathscr{F}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G}: {}_B\mathscr{C} \to {}_A\mathscr{C}$ such that $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{C}$-functors.

Without loss of generality we shall only consider equivalences $\mathscr{F}$ and $\mathscr{G}$ together with isomorphisms $\Phi: \mathscr{F}\mathscr{G} \cong Id$ and $\Psi: \mathscr{G}\mathscr{F} \cong Id$ such that $\mathscr{F}\Psi = \Phi\mathscr{F}$ and $\Psi\mathscr{G} = \mathscr{G}\Phi$. Then Φ and Ψ and their inverses are already adjointness morphisms.

5.1. Theorem. *Let $\mathscr{C}$ be an arbitrary monoidal category. Let $\mathscr{F}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ and $\mathscr{G}: {}_B\mathscr{C} \to {}_A\mathscr{C}$ be inverse $\mathscr{C}$-equivalences. Then there are objects $P \in {}_A\mathscr{C}_B$ and $Q \in {}_B\mathscr{C}_A$ such that*

a) *there are natural isomorphisms*

$$\mathscr{F}(M) \cong Q \otimes_A M \cong {}_A[P, M] \quad in \quad {}_A\mathscr{C},$$

$$\mathscr{G}(N) \cong P \otimes_B N \cong {}_B[Q, M] \quad in \quad {}_B\mathscr{C},$$

and P B-coflat and Q A-coflat.

b) *there are isomorphisms of $A-A$-resp. $B-B$-biobjects*

$$A \cong P \otimes_B Q \quad and \quad B \cong Q \otimes_A P$$

such that the diagrams

$$\begin{array}{ccc} P \otimes (Q \otimes P) \cong (P \otimes Q) \otimes P & \longrightarrow & A \otimes P \\ \downarrow & & \downarrow \\ P \otimes B & \longrightarrow & P \end{array}$$

and

$$\begin{array}{ccc} Q\otimes(P\otimes Q) \cong (Q\otimes P)\otimes Q & \longrightarrow & B\otimes Q \\ \downarrow & & \downarrow \\ Q\otimes A & \longrightarrow & Q \end{array}$$

commute,

c) *there are isomorphisms*

$$_B[Q, B] \cong P \quad in \quad {}_A\mathscr{C}_B,$$

$$_A[P, A] \cong Q \quad in \quad {}_B\mathscr{C}_A,$$

d) *there are isomorphisms*

$$_B[Q, Q] \cong A \quad in \quad {}_A\mathscr{C}_A \quad \textit{and as monoids,}$$

$$_A[P, P] \cong B \quad in \quad {}_B\mathscr{C}_B \quad \textit{and as monoids.}$$

PROOF. By the symmetry of the situation we only have to prove one half of the assertions.

There is an isomorphism $\mathscr{F}(M)\cong Q\otimes_A M$ natural in M by Theorem 4.2 since $\mathscr{F}$ is a $\mathscr{C}$-functor and clearly preserves difference cokernels as an equivalence. By the same theorem we have $\mathscr{G}(N)\cong {}_B[Q, N]$ natural in N, since $\mathscr{G}$ is adjoint to $\mathscr{F}$. This proves a).

We have an isomorphism $A\cong\mathscr{G}\mathscr{F}(A)\cong P\otimes_B(Q\otimes_A A)\cong P\otimes_B Q$ in ${}_A\mathscr{C}$. Furthermore we have a commutative diagram

$$\begin{array}{ccccc} A\otimes A & \cong & P\otimes_B(Q\otimes_A(A\otimes A)) & \cong & P\otimes_B(Q\otimes A) \\ \downarrow & & \downarrow & & \downarrow \\ A & \cong & P\otimes_B(Q\otimes_A A) & \cong & P\otimes_B Q \end{array}$$

hence $A\cong P\otimes_B Q$ as $A-A$-biobjects.

The adjunction morphism $\Psi: \mathscr{G}\mathscr{F}\cong Id$ induces the evaluation morphism $\Psi': P\otimes_{B\,A}[P, M]\cong M$ with $\Psi'(p\otimes f)=\langle p\rangle f$. By definition of the isomorphism $A\cong P\otimes_B Q$ we get a commutative diagram

$$\begin{array}{ccccc} P\otimes_B(Q\otimes_A M) & \cong & (P\otimes_B Q)\otimes_A M & \cong & A\otimes_A M \\ & \searrow & & \swarrow & \\ & P\otimes_{B\,A}[P, M] & \xrightarrow{\Psi'} & M & \end{array}$$

Hence if the isomorphism $P\otimes_B Q\cong A$ is described by $p\otimes_B q\mapsto pq$ and the morphism $Q\otimes_A M\mapsto {}_A[P, M]$ is given by

$$q\otimes_A m \mapsto \varphi(q\otimes_A m), \quad \text{we get} \quad (pq)m = \langle p\rangle\varphi(q\otimes_A m).$$

Now if $M\in {}_A\mathscr{C}_B$ then we get $\langle p\rangle\varphi(q\otimes_A mb)=(pq)(mb)=((pq)m)b=(\langle p\rangle\varphi(q\otimes_A m))b=$ $=\langle p\rangle(\varphi(q\otimes_A m)b)$, hence $Q\otimes_A M$ and ${}_A[P, M]$ are isomorphic as $B-B$-biobjects.

In particular we get ${}_A[P, A]\cong Q$ as $B-A$-biobjects and ${}_A[P, P]\cong Q\otimes_A P\cong B$ as $B-B$-biobjects.

To prove the monoid isomorphisms we first observe that

$$\begin{array}{ccc} P\otimes(Q\otimes P)\cong(P\otimes Q)\otimes P & \longrightarrow & A\otimes P \quad \text{and} \\ \downarrow & & \downarrow \\ P\otimes B & \longrightarrow & P \end{array}$$

$$\begin{array}{ccc} Q\otimes(P\otimes Q)\cong(Q\otimes P)\otimes Q & \longrightarrow & B\otimes Q \\ \downarrow & & \downarrow \\ Q\otimes A & \longrightarrow & Q \end{array}$$

commute. This follows from $\mathscr{F}\cong Q\otimes_A$, $\mathscr{G}\cong P\otimes_B$ and from the fact that $\Phi\mathscr{F}\colon \mathscr{F}\mathscr{G}\mathscr{F}\to\mathscr{F}$ and $\mathscr{F}\Psi\colon \mathscr{F}\mathscr{G}\mathscr{F}\to\mathscr{F}$ resp. $\mathscr{G}\Phi\colon \mathscr{G}\mathscr{F}\mathscr{G}\to\mathscr{G}$ and $\Psi\mathscr{G}\colon\mathscr{G}\mathscr{F}\mathscr{G}\to\mathscr{G}$ are equal. So we get

$$\langle p\rangle\varphi(q'\otimes p')\varphi(q''\otimes p'')=(pq')(p'q'')p''=p(q'p')(q''p'').$$

Since the isomorphism ${}_A[P, P]\cong B$ is given by

$${}_A[P, P]\xleftarrow{\ \varphi\ } Q\otimes_A P\cong B$$

or $\varphi(q'\otimes p')\mapsto q'p'$, the composition $\varphi(q'\otimes p')\varphi(q''\otimes p'')$ is mapped to the product $(q'p')(q''p'')$. If φ $(q'\otimes p')$ is the identity then $p(q'p')=p$ for all p. But ${}_A[P, P]\to B$ is an isomorphism hence $q'p'=1\in B$.

5.2. Corollary: *The morphisms*

$$P(X)\times{}_A[P, A](Y)\ni(p, f)\mapsto\langle p\rangle f\in A(X\otimes Y)$$

and

$${}_A[P, A](X)\times P(Y)\ni(f, p)\mapsto fp\in{}_A[P, P](X\otimes Y)$$

with $\langle p'\rangle(fp):=(\langle p'\rangle f)p$ *induce isomorphisms*

$$P\otimes_B{}_A[P, A]\cong A \quad \textit{and} \quad {}_A[P, A]\otimes_A P\cong{}_A[P, P].$$

The analogous assertions hold for Q and B.

PROOF. The first isomorphism, the evaluation morphism, was discussed in the proof of 5.1. The second isomorphism is just given by

$${}_A[P, A]\otimes_A P\xrightarrow{\ \varphi\otimes P\ } Q\otimes_A P\xrightarrow{\ \varphi\ }{}_A[P, P].$$

We have seen that each $\mathscr{C}$-equivalence is induced by some object $P\in{}_A\mathscr{C}_B$ with the properties of Corollary 5.2. The converse will proved after a more detailed study of the properties exhibited in Corollary 5.2.

An object $P\in{}_A\mathscr{C}$ will be called *finite*, if ${}_A[P, A]$ and $B:={}_A[P, P]$ exist, if P is B-coflat and ${}_A[P, A]$ is A-coflat and if the morphism ${}_A[P, A]\otimes_A P\to{}_A[P, P]$ induced by ${}_A[P, A](X)\times P(Y)\ni(f, p)\mapsto fp\in{}_A[P, P](X\otimes Y)$ with $\langle p'\rangle fp=(\langle p'\rangle f)p$ is an isomorphism. P will be called *faithfully projective* if it is finite and if the morphism $P\otimes_B{}_A[P, A]\to A$ induced by the evaluation is also an isomorphism.

5.3. Theorem. *Let A, B be monoids in $\mathscr{C}$, $P \in {}_A\mathscr{C}_B$ B-coflat and $Q \in {}_B\mathscr{C}_A$ A-coflat. Given morphisms $f\colon P \otimes_B Q \to A$ in ${}_A\mathscr{C}_A$ and $g\colon Q \otimes_A P \to B$ in ${}_B\mathscr{C}_B$ such that the diagrams*

$$\begin{array}{ccc} P \otimes_B (Q \otimes_A P) \cong (P \otimes_B Q) \otimes_A P & \xrightarrow{f \otimes_A P} & A \otimes_A P \\ \downarrow{\scriptstyle P \otimes_B g} & & \downarrow \\ P \otimes_B B & \longrightarrow & P \end{array}$$

$$\begin{array}{ccc} Q \otimes_A (P \otimes_B Q) \cong (Q \otimes_A P) \otimes_B Q & \xrightarrow{g \otimes_B Q} & B \otimes_B Q \\ \downarrow{\scriptstyle Q \otimes_A f} & & \downarrow \\ Q \otimes_A A & \longrightarrow & Q \end{array}$$

commute. Assume that there is $p_0 \otimes_B q_0 \in P \otimes_B Q(I)$ such that $p_0 q_0 := f(p_0 \otimes_B q_0) = 1 \in A(I)$. Then f is an isomorphism. Assume that in addition there is $q_1 \otimes_A p_1 \in Q \otimes_A P(I)$ such that $q_1 p_1 := g(q_1 \otimes_A p_1) = 1 \in B(I)$. Then $P \otimes_B\colon {}_B\mathscr{C} \to {}_A\mathscr{C}$ and $Q \otimes_A\colon {}_A\mathscr{C} \to {}_B\mathscr{C}$ are inverse $\mathscr{C}$-equivalences. In particular $P \in {}_A\mathscr{C}$ and $Q \in {}_B\mathscr{C}$ are faithfully projective.

Proof. Define $f'\colon A \to P \otimes_B Q$ by $f'(a) = ap_0 \otimes_B q_0$. Then $ff'(a) = ap_0 q_0 = a$ and $f'f(p \otimes_B q) = (pq)p_0 \otimes_B q_0 = p(qp_0) \otimes_B q_0 = p \otimes_B (qp_0) q_0 = p \otimes_B q(p_0 q_0) = p \otimes_B q$. Hence f is an isomorphism.

Furthermore the functors $P \otimes_B Q \otimes_A \cong A \otimes_A$ and $Q \otimes_A P \otimes_B \cong B \otimes_B$ are both isomorphic to the identity-functors on ${}_A\mathscr{C}$ resp. ${}_B\mathscr{C}$, hence they are inverse equivalences. Furthermore $P \otimes_B$ and $Q \otimes_A$ are $\mathscr{C}$-functors by Theorem 4.2.

5.4. Theorem. *Let $P \in {}_A\mathscr{C}$ be faithfully projective. Then ${}_A[P, -]\colon {}_A\mathscr{C} \to {}_{A[P,P]}\mathscr{C}$ exists and is a $\mathscr{C}$-equivalence.*

Proof. By definition ${}_A[P, A]$ and ${}_A[P, P] = B$ exist. Furthermore $P \in {}_A\mathscr{C}_B$, $Q := {}_A[P, A] \in {}_B\mathscr{C}_A$ and the hypotheses of Theorem 5.3. are satisfied by the very definition of $Q \otimes_A P \to B$ and $P \otimes_B Q \to A$. So $Q \otimes_A\colon {}_A\mathscr{C} \to {}_B\mathscr{C}$ is a $\mathscr{C}$-equivalence. By Theorem 5.1. we get $Q \otimes_A \cong {}_A[P, -]$.

Let us now apply our theorems to the case where the tensor-product in $\mathscr{C}$ is the (direct) product (example c) of § 1). Furthermore assume that each canonical epimorphism $M \times N \to M \times_A N$ induces a surjective map $M \times N(I) \to M \times_A N(I)$. This is for example the case if I is projective in the category $\mathscr{C}$. We say that $M \times N \to M \times_A N$ is *rationally surjective*. Assume that ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$ are $\mathscr{C}$-equivalent by $P \otimes_B -\colon {}_B\mathscr{C} \to {}_A\mathscr{C}$ and $Q \otimes_A\colon {}_A\mathscr{C} \to {}_B\mathscr{C}$. Then we have surjective maps $f\colon P(I) \times Q(I) \cong P \times Q(I) \to A(I)$ and $g\colon Q(I) \times P(I) \cong Q \times P(I) \to B(I)$ such that $(pq)p' = p(qp')$ and $(qp)q' = q(pq')$ if $f(p, q) = pq$ and $g(q, p) = qp$. Let $p_i \in P(I)$, $q_i \in Q(I)$, $i = 0, 1$ be chosen such that $p_0 q_0 = 1 \in A(I)$, $q_1 p_1 = 1 \in B(I)$.

Let us now assume that each element in $A(I)$ which has a left inverse has also a right inverse. We wish to show $A \cong B$ as monoids. First we show $p_1 q_1 = 1 \in A$. By definition we have $(p_0 q_1)(p_1 q_0) = p_0 (q_1 p_1) q_0 = p_0 q_0 = 1 \in A(I)$, hence $(p_1 q_0)(p_0 q_1) = 1$. Furthermore we have $p_0 q_1 p_1 = p_0$. This implies $p_1 q_1 = 1 \cdot p_1 q_1 = (p_1 q_0 p_0 q_1) p_1 q_1 = p_1 q_0 (p_0 q_1 p_1) q_1 = p_1 q_0 p_0 q_1 = 1 \in A(I)$. Now define morphisms $P(X) \ni p \mapsto pq_1 \in A(X)$ and $A(X) \ni a \mapsto ap_1 \in P$. They are obviously mutually inverse morphisms in ${}_A\mathscr{C}$. Hence $B \cong {}_A[P, P] \cong {}_A[A, A] \cong A$.

As a special case we get

5.5. Corollary: *In the category of sets $\mathscr{S}$ with the product as monoidal category let A be a group or a commutative monoid or finite. Then ${}_A\mathscr{S}$ and ${}_B\mathscr{S}$ are equivalent iff $A \cong B$.*

PROOF. In $\mathscr{S}$ the morphism-sets form an inner hom-functor, so by Theorem 4.3. each equivalence ${}_A\mathscr{S} \cong {}_B\mathscr{S}$ is an $\mathscr{S}$-equivalence. Furthermore $\{\emptyset\}$ is projective in $\mathscr{S}$. If A is a group or commutative or finite then each element which has a left inverse in $A(I)$ has also a right inverse. So all conditions of the previous discussion are satisfied. Hence $A \cong B$. The converse is trivial.

The central part of the Morita Theorems

For this section we will always assume that $\mathscr{C}$ is a symmetric monoidal category. Let us consider $\mathscr{C}$-functors $\mathscr{U}, \mathscr{V}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ such that there are $P, Q \in {}_B\mathscr{C}_A$ with $\mathscr{C}$-isomorphisms $\mathscr{U} \cong P\otimes_A$, $\mathscr{V} \cong Q\otimes_A$.

Define a new $\mathscr{C}$-functor $\mathscr{U}\otimes Y$ for $Y \in \mathscr{C}$ by $\mathscr{U}\otimes Y(M) := \mathscr{U}(M\otimes Y) \cong \cong P\otimes_A(M\otimes Y)$. Because of the symmetry of $\mathscr{C}$ we have $\mathscr{U}\otimes Y(M) \cong (P\otimes Y)\otimes_A M$ hence $\mathscr{U}\otimes Y$ indeed is a $\mathscr{C}$-functor.

Define $[\mathscr{U}, \mathscr{V}](Y) := \mathscr{C}\text{-Mor}(\mathscr{U}\otimes Y, \mathscr{V})$ as the set of $\mathscr{C}$-morphisms from $\mathscr{U}\otimes Y$ to $\mathscr{V}$. For $h: Z \to Y$ in $\mathscr{C}$ define $[\mathscr{U}, \mathscr{V}](h): [\mathscr{U}, \mathscr{V}](Y) \to [\mathscr{U}, \mathscr{V}](Z)$ by $[\mathscr{U}, \mathscr{V}](h)(\varphi)(M) := \big(\mathscr{U}(M\otimes Z) \xrightarrow{\mathscr{U}(M\otimes h)} \mathscr{U}(M\otimes Y) \xrightarrow{\varphi(M)} \mathscr{V}(M)\big)$. Then $[\mathscr{U}, \mathscr{V}]$ is a contravariant functor from $\mathscr{C}$ to the category of sets.

6.1. Theorem. *There is a natural isomorphism of functors from $\mathscr{C}$ to the category of sets:*

$$[\mathscr{U}, \mathscr{V}](Y) \cong {}_B\mathscr{C}_A(P\otimes Y, Q).$$

If ${}_B[P, Q]_A$ exists then $[\mathscr{U}, \mathscr{V}](Y) \cong {}_B[P, Q]_A(Y)$.

PROOF. Let $f \in {}_B\mathscr{C}_A(P\otimes Y, Q)$. Then define $\varphi \in [\mathscr{U}, \mathscr{V}](Y)$ by

$$\varphi(M) := \big(\mathscr{U}(M\otimes Y) \cong (P\otimes Y)\otimes_A M \xrightarrow{f\otimes_A M} Q\otimes_A M \cong \mathscr{V}(M)\big).$$

For $g \in {}_A\mathscr{C}(M, N)$ we get a commutative diagram

$$\begin{array}{ccccccc}
\mathscr{U}(M\otimes Y) & \cong & (P\otimes Y)\otimes_A M & \xrightarrow{f\otimes_A M} & Q\otimes_A M & \cong & \mathscr{V}(M) \\
\downarrow{\scriptstyle \mathscr{U}(g\otimes Y)} & & \downarrow{\scriptstyle (P\otimes Y)\otimes_A g} & & \downarrow{\scriptstyle Q\otimes_A g} & & \downarrow{\scriptstyle \mathscr{V}(g)} \\
\mathscr{U}(N\otimes Y) & \cong & (P\otimes Y)\otimes_A N & \xrightarrow{f\otimes_A N} & Q\otimes_A N & \cong & \mathscr{V}(N)
\end{array}$$

hence φ is a natural transformation from $\mathscr{U}\otimes Y$ to $\mathscr{V}$. Furthermore the following diagram commutes:

$$\begin{array}{ccccccc}
\mathscr{U}((M\otimes X)\otimes Y) & \cong & (P\otimes Y)\otimes_A(M\otimes X) & \xrightarrow{f\otimes_A(M\otimes X)} & Q\otimes_A(M\otimes X) & \cong & \mathscr{V}(M\otimes X) \\
\wr\| & & \wr\| & & \wr\| & & \wr\| \\
\mathscr{U}(M\otimes Y)\otimes X & \cong & ((P\otimes Y)\otimes_A M)\otimes X & \xrightarrow{(f\otimes_A M)\otimes X} & (Q\otimes_A M)\otimes X & \cong & \mathscr{V}(M)\otimes X
\end{array}$$

Hence φ is a $\mathscr{C}$-morphism. This defines a map

$$\Sigma : {}_B\mathscr{C}_A(P\otimes Y, Q) \to [\mathscr{U}, \mathscr{V}](Y).$$

Conversely let $\varphi\in[\mathscr{U},\mathscr{V}](Y)$ and define $f\in {}_B\mathscr{C}_A(P\otimes Y, Q)$ by $f:= (P\otimes Y\cong P\otimes_A A\otimes Y\cong \mathscr{U}(A\otimes Y)\xrightarrow{\varphi(A)} \mathscr{V}(A)\cong Q\otimes_A A\cong Q)$. Clearly $f\in {}_B\mathscr{C}(P\otimes Y, Q)$. To show that $f\in {}_B\mathscr{C}_A(P\otimes Y, Q)$ consider the following commutative diagram

$$\begin{array}{ccccccccccc}
P\otimes_A(A\otimes Y)\otimes A & \cong & ((P\otimes Y)\otimes_A A)\otimes A & \cong & \mathcal{U}(A\otimes Y)\otimes A & \xrightarrow{\varphi(A)\otimes A} & \mathcal{V}(A)\otimes A & \cong & (Q\otimes_A A)\otimes A & & \\
\wr\!\! \| & & \wr\!\! \| & & \wr\!\! \| & & \wr\!\! \| & & \wr\!\! \| & & \wr\!\! \| \\
(P\otimes Y)\otimes A & \cong & (P\otimes Y)\otimes_A(A\otimes A) & \cong & \mathcal{U}((A\otimes A)\otimes Y) & \xrightarrow{\varphi(A\otimes A)} & \mathcal{V}(A\otimes A) & \cong & Q\otimes_A(A\otimes A) & \cong & Q\otimes A \\
\downarrow{\scriptstyle \nu_{P\otimes Y}} & & \downarrow{\scriptstyle (P\otimes Y)\otimes_A \mu_A} & & \downarrow{\scriptstyle \mathcal{U}(\mu_A\otimes Y)} & & \downarrow{\scriptstyle \mathcal{V}(\mu_A)} & & \downarrow{\scriptstyle Q\otimes_A \mu_A} & & \downarrow{\scriptstyle \nu_Q} \\
P\otimes Y & \cong & (P\otimes Y)\otimes_A A & \cong & \mathcal{U}(A\otimes Y) & \xrightarrow{\varphi(A)} & \mathcal{V}(A) & \cong & Q\otimes_A A & \cong & Q
\end{array}$$

where the morphism from $(P\otimes Y)\otimes A$ to $Q\otimes A$ along the upper side of the diagram is just $f\otimes A$ and the morphism from $P\otimes Y$ to Q along the lower side is f. Hence f is a right A-morphism. So we have a map

$$\Pi : [\mathscr{U}, \mathscr{V}](Y) \to {}_B\mathscr{C}_A(P\otimes Y, Q).$$

Now

$$\Pi\Sigma(f) = \big(P\otimes Y \cong \mathscr{U}(A\otimes Y) \xrightarrow{\Sigma(f)(A)} \mathscr{V}(A) \cong Q\big) =$$
$$= \big(P\otimes Y \cong (P\otimes Y)\otimes_A A \xrightarrow{f\otimes_A A} Q\otimes_A A \cong Q\big) = f$$

and

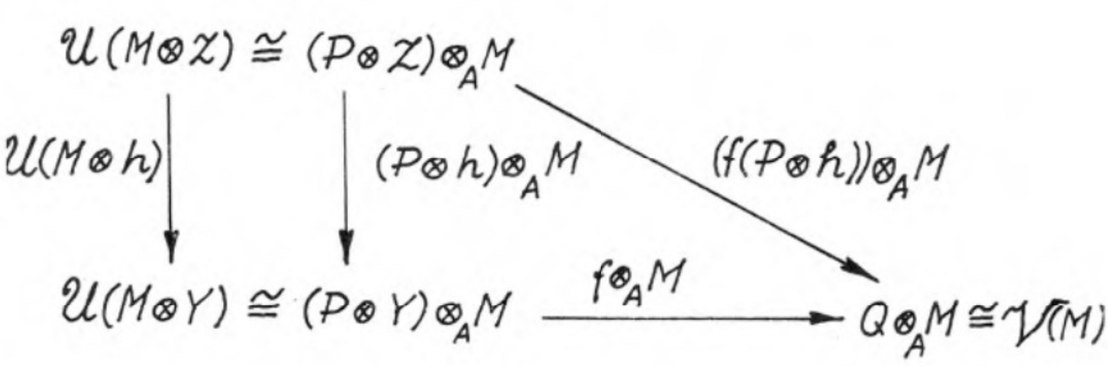
$$\Sigma\Pi(\varphi)(M) = \big(\mathscr{U}(M\otimes Y) \cong (P\otimes Y)\otimes_A M \xrightarrow{\pi(\varphi)\otimes_A M} Q\otimes_A M \cong \mathscr{V}(M)\big) =$$
$$= \big(\mathscr{U}(M\otimes Y) \cong \mathscr{U}(A\otimes Y)\otimes_A M \xrightarrow{\varphi(A)\otimes_A M} \mathscr{V}(A)\otimes_A M \cong \mathscr{V}(M)\big) = \varphi(M)$$

since φ is a $\mathscr{C}$-morphism. Hence we have $[\mathscr{U},\mathscr{V}](Y)\cong {}_B\mathscr{C}_A(P\otimes Y, Q)$.

It remains to show that this isomorphism is a natural transformation. Let $h: Z\to Y$ be in $\mathscr{C}$. Then

$$\begin{array}{ccc}
{}_B\mathscr{C}_A(P\otimes Y, Q) & \xrightarrow{\Sigma(Y)} & [\mathscr{U},\mathscr{V}](Y) \\
\downarrow{\scriptstyle {}_B\mathscr{C}_A(P\otimes h, Q)} & & \downarrow{\scriptstyle [\mathscr{U},\mathscr{V}](h)} \\
{}_B\mathscr{C}_A(P\otimes Z, Q) & \xrightarrow{\Sigma(Z)} & [\mathscr{U},\mathscr{V}](Z)
\end{array}$$

commutes since for $f \in {}_B\mathscr{C}_A(P \otimes Y, Q)$ we have



commutative and thus

$$([\mathscr{U}, \mathscr{V}](h) \circ \Sigma(Y))(f)(M) =$$

$$= (\mathscr{U}(M \otimes Z) \xrightarrow{\mathscr{U}(M \otimes h)} \mathscr{U}(M \otimes Y) \cong (P \otimes Y) \otimes_A M \xrightarrow{f \otimes_A M} Q \otimes_A M \cong \mathscr{V}(M)) =$$

$$= (\mathscr{U}(M \otimes Z) \cong (P \otimes Z) \otimes_A M \xrightarrow{(f(P \otimes h)) \otimes_A M} Q \otimes_A M \cong \mathscr{V}(M)) =$$

$$= (\Sigma(Z) \circ {}_B\mathscr{C}_A(P \otimes h, Q))(f)(M).$$

For this theorem we have two applications. Before we discuss them, we have to introduce the notion of the center of a monoid.

It is clear that ${}_A\mathscr{C}(A, A) \cong A(I)$ as monoids in the category of sets. The isomorphism is given by

$${}_A\mathscr{C}(A, A) \ni f \mapsto f(1) = f\eta \in A(I)$$

$$A(I) \ni a \mapsto (A(X) \ni b \mapsto ba \in A(X)) \in {}_A\mathscr{C}(A, A).$$

Now those elements $a \in A(I)$ which commute with all $b \in A(X)$ for all X induce in ${}_A\mathscr{C}(A, A)$ precisely the $A-A$-morphisms ${}_A\mathscr{C}_A(A, A)$, which then is a commutative monoid. So a possible definition of the center of A could be ${}_A\mathscr{C}_A(A, A)$. But this is only a set, not an object in $\mathscr{C}$. A possible generalization to an object in $\mathscr{C}$ is ${}_A[A, A]_A$ if this exists. If it does not exist we know at least the functor represented by this object. So we define the *center of A* as a functor from $\mathscr{C}$ to $\mathscr{S}$, the category of sets, by Cent $(A)(X) := {}_A\mathscr{C}_A(A \otimes X, A)$. If ${}_A[A, A]_A$ exists we have Cent $(A)(X) \cong$ $\cong {}_A[A, A]_A(X)$.

As in § 3 Cent (A) can only be defined in a symmetric monoidal category in contrast to ${}_A\mathscr{C}_A(A, A)$. In § 2 we showed that ${}_I\mathscr{C} \cong \mathscr{C}$ and $\mathscr{C} \cong \mathscr{C}_I$ hence $\mathscr{C}(I, I) \cong$ $\cong {}_I\mathscr{C}_I(I, I)$ is a commutative monoid [18, Theorem 1] for (possibly nonsymmetric) monoidal categories.

Let A be a monoid in a symmetric monoidal category $\mathscr{C}$. Let ${}_AId: {}_A\mathscr{C} \to {}_A\mathscr{C}$ denote the identity functor. Then ${}_AId$ is clearly a $\mathscr{C}$-functor and ${}_AId \cong A \otimes_A$ as $\mathscr{C}$-functors.

6.2. Theorem. $\mathrm{Cent}\,(A) \cong [{}_A Id, {}_A Id]$.

PROOF. $\mathrm{Cent}\,(A)(X) = {}_A\mathscr{C}_A(A \otimes X, A) \cong [{}_A Id, {}_A Id](X)$.

The isomorphism of Theorem 6.2. is only an isomorphism of objects in $\mathscr{C}$. But there is an additional structure, a multiplication on these functors. If they were representable the representing objects in $\mathscr{C}$ would be monoids. The multiplicative structure on ${}_A[A, A]_A$ as it has been studied in § 3 is reflected in ${}_A\mathscr{C}_A(A \otimes X, A)$ by the commutative diagram

$$\begin{array}{ccc} {}_A[A, A]_A(X) \times {}_A[A, A]_A(Y) & \to & {}_A[A, A]_A(X \otimes Y) \\ \wr\| & & \wr\| \\ {}_A\mathscr{C}_A(A \otimes X, A) \times {}_A\mathscr{C}_A(A \otimes Y, A) & \to & {}_A\mathscr{C}_A(A \otimes X \otimes Y, A) \end{array}$$

where the lower map is given by

$$(f, g) \mapsto (A \otimes X \otimes Y \xrightarrow{f \otimes Y} A \otimes Y \xrightarrow{g} A).$$

The unit is described by

$$I(X) \ni f \mapsto (A \otimes X \xrightarrow{A \otimes f} A \otimes I \cong A) \in {}_A\mathscr{C}_A(A \otimes X, A).$$

$[{}_A Id, {}_A Id]$ carries a multiplaciticve structure via

$$[{}_A Id, {}_A Id](X) \times [{}_A Id, {}_A Id](Y) \xrightarrow{T} [{}_A Id, {}_A Id](X \otimes Y)$$

by
$$T(\varphi, \psi) = ({}_A Id \otimes X \otimes Y \xrightarrow{\varphi \otimes Y} {}_A Id \otimes Y \xrightarrow{\varphi} {}_A Id)$$

and there is a unit

$$I(X) \ni f \mapsto ({}_A Id \otimes X \xrightarrow{{}_A Id \otimes f} {}_A Id \otimes I \cong {}_A Id) \in [{}_A Id, {}_A Id](X).$$

Using the isomorphism of Theorem 6.1. it is easy to see that they are compatible with the multiplication and the unit map. Hence the isomorphism of Theorem 6.2. is a „monoid isomorphism".

6.3. Corollary: *Let ${}_A\mathscr{C}$ and ${}_B\mathscr{C}$ be $\mathscr{C}$-equivalent. Then* $\mathrm{Cent}\,(A) \cong \mathrm{Cent}\,(B)$ *as functors from $\mathscr{C}$ to $\mathscr{S}$. If both functors are representable then the two representing objects are isomorphic as commutative monoids in $\mathscr{C}$:*

$$_A[A, A]_A \cong {}_B[B, B]_B.$$

PROOF. We show $[{}_A Id, {}_A Id] \cong [{}_B Id, {}_B Id]$. Let $\mathscr{F}: {}_A\mathscr{C} \to {}_B\mathscr{C}$ the given $\mathscr{C}$-equivalence. First we show

$$[{}_A Id, {}_A Id](X) \cong [\mathscr{F}, \mathscr{F}](Y),$$

or

$\mathscr{C}\text{-Mor}\,({}_A Id \otimes Y, {}_A Id) \cong \mathscr{C}\text{-Mor}\,(\mathscr{F} \otimes Y, \mathscr{F})$.

Let $\varphi : {}_A Id \otimes Y \to {}_A Id$ be a natural transformation. Define $\mathscr{F} \circ \varphi : \mathscr{F} \otimes Y \to \mathscr{F}$ by $\mathscr{F} \circ \varphi(M) : \mathscr{F}(M \otimes Y) \to \mathscr{F}(M)$ as $\mathscr{F} \circ \varphi(M) = \mathscr{F}(\varphi(M))$. Since $\mathscr{F}$ is an equivalence it is clear that $\varphi \mapsto \mathscr{F} \circ \varphi$ is a bijection between the sets of natural transformations. Now we show that φ is a $\mathscr{C}$-morphism iff $\mathscr{F} \circ \varphi$ is a $\mathscr{C}$-morphism. φ is a $\mathscr{C}$-morphism iff the diagrams

$$\begin{array}{ccc} M\otimes X\otimes Y & \xrightarrow{\varphi(M\otimes X)} & M\otimes X \\ \wr\| & & \wr\| \\ M\otimes Y\otimes X & \xrightarrow{\varphi(M)\otimes X} & M\otimes X \end{array}$$

commute. $\mathscr{F}\circ\varphi$ is a $\mathscr{C}$-morphism iff the outer diagrams of

$$\begin{array}{ccc} \mathscr{F}(M\otimes X\otimes Y) & \xrightarrow{\mathscr{F}\circ\varphi(M\otimes X)} & \mathscr{F}(M\otimes X) \\ \wr\| & & \wr\| \\ \mathscr{F}((M\otimes Y)\otimes X) & \xrightarrow{\mathscr{F}(\varphi(M)\otimes X)} & \mathscr{F}(M\otimes X) \\ \wr\| & & \wr\| \\ \mathscr{F}(M\otimes Y)\otimes X & \xrightarrow{\mathscr{F}\circ\varphi(M)\otimes X} & \mathscr{F}(M)\otimes X \end{array}$$

commute where the lower part commutes in any case since $\mathscr{F}$ is a $\mathscr{C}$-functor. But the upper part commutes iff the previous diagram commutes. Hence

$$\mathscr{C}-\mathrm{Mor}\,({}_AId\otimes Y,\ {}_AId)\cong\mathscr{C}-\mathrm{Mor}\,(\mathscr{F}\otimes Y,\mathscr{F}).$$

Now we show $[{}_BId,\ {}_BId](Y)\cong[\mathscr{F},\mathscr{F}](Y)$ or

$$\mathscr{C}-\mathrm{Mor}\,({}_BId\otimes Y,\ {}_BId)\cong\mathscr{C}-\mathrm{Mor}\,(\mathscr{F}\otimes Y,\mathscr{F}).$$

It is clear that the correspondence between $\varphi\colon {}_BId\otimes Y\to {}_BId$ and $\varphi\mathscr{F}\circ\colon \mathscr{F}\otimes Y\to\mathscr{F}$ with

$$(\varphi\circ\mathscr{F})(M):=\big(\mathscr{F}(M\otimes Y)\cong\mathscr{F}(M)\otimes Y\xrightarrow{\varphi(\mathscr{F}(M))}\mathscr{F}(M)\big)$$

induces an isomorphism between the sets of natural transformation, since $\mathscr{F}$ is an equivalence. Furthermore φ is a $\mathscr{C}$-morphism iff the diagrams

$$\begin{array}{ccc} N\otimes X\otimes Y & \xrightarrow{\varphi(N\otimes X)} & N\otimes X \\ \wr\| & & \| \\ N\otimes Y\otimes X & \xrightarrow{\varphi(N)\otimes X} & N\otimes X \end{array}$$

commute. On the other hand $\varphi\circ\mathscr{F}$ is a $\mathscr{C}$-morphism iff the outer diagrams

$$\begin{array}{ccc}
\mathscr{F}(M\otimes X\otimes Y) & \xrightarrow{\varphi\circ\mathscr{F}(M\otimes X)} & \mathscr{F}(M\otimes X) \\
\wr\| & & \wr\| \\
\mathscr{F}(M\otimes X)\otimes Y & \xrightarrow{\varphi(\mathscr{F}(M\otimes X))} & \mathscr{F}(M\otimes X) \\
\wr\| & & \wr\| \\
\mathscr{F}(M)\otimes Y\otimes X & \xrightarrow{\varphi(\mathscr{F}(M)\otimes X} & \mathscr{F}(M)\otimes X \\
\wr\| & & \wr\| \\
\mathscr{F}(M\otimes Y)\otimes X & \xrightarrow{\varphi\circ\mathscr{F}(M)\otimes X} & \mathscr{F}(M)\otimes X
\end{array}$$

commute. The first and third part commute by definition. In the middle part take into account that $\mathscr{F}$ is a $\mathscr{C}$-functor. Then it commutes iff the previous diagram for φ commutes. Hence $[{}_BId, {}_BId](Y)\cong[\mathscr{F},\mathscr{F}](Y)\cong[{}_AId, {}_AId](Y)$.

The reader can easily verify that these isomorphisms are natural isomorphisms in Y. Furthermore they preserve the „multiplication" given by composition of morphisms just before Corollary 6.3. They also preserve the „unit". Hence

$$_A[A, A]_A \cong {}_B[B, B]_B$$

as monoids (if they exist) or

$$\mathrm{Cent}\,(A) \cong \mathrm{Cent}\,(B)$$

with the multiplicative structure.

6.4. Corollary: *Let $\mathscr{U}\colon {}_A\mathscr{C}\to\mathscr{C}$ be the underlying functor. Then $[\mathscr{U},\mathscr{U}](Y)\cong$ $\cong A^{\mathrm{op}}(Y)$ natural in $Y\in\mathscr{C}$ and compatible with the multiplication on both sides.*

Proof. By Theorem 6.1. and the fact $\mathscr{U}\cong A\otimes_A$ we have $[\mathscr{U},\mathscr{U}](Y)\cong$ $\cong\mathscr{C}_A(A\otimes Y, A)\cong A^{\mathrm{op}}(Y)$ as left multiplications and these isomorphisms are natural in Y and compatible with the multiplication.

So we have seen that just from the knowledge of the underlying functor $\mathscr{U}$ we may regain the monoid A up to an isomorphism.

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