A fixed point theorem for compact metric spaces

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In a recent paper, see [1], the following theorem was proved:

Theorem 1. *If T is a mapping of the complete metric space X into itself satisfying the condition*

$$d(Tx, Ty) \leqq c[d(x, Ty)+d(y, Tx)]$$

for all x, y in X, where $0 \leqq c < 1/2$, then T has a unique fixed point.

We will now prove the following theorem:

Theorem 2. *If T is a continuous mapping of a compact metric space X into itself satisfying the condition*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Ty)+d(y, Tx)]$$

for all distinct x, y in X, then T has a unique fixed point.

PROOF. Define a real-valued function f on X by

$$f(x) = d(x, Tx).$$

Since d and T are continuous functions it follows that f is a continuous function on X. Since X is compact it achieves its minimum value and so there exists a point z in X such that

$$f(z) = \inf\{f(x) \colon x \in X\}.$$

We will now suppose that $Tz \neq z$. Then by hypothesis

$$d(Tz, T^2 z) < \frac{1}{2}[d(z, T^2 z)+d(Tz, Tz)]$$

$$= \frac{1}{2} d(z, T^2 z) \leqq \frac{1}{2}[d(z, Tz)+d(Tz, T^2 z)]$$

and so

$$d(Tz, T^2 z) < d(z, Tz)$$

or equivalently

$$f(Tz) < f(z).$$

This contradicts the definition of z and so we must have $Tz = z$. Thus z is a fixed point of T.

We will now prove that z is unique. Suppose that z' is a second fixed point with $z \neq z'$. Then

$$d(z, z') = d(Tz, Tz') < \frac{1}{2}[d(z, Tz')+d(z', Tz)] = d(z, z'),$$

giving a contradiction and so proves the uniqueness of z. This completes the proof of the theorem.

We will now show from the following *example* that the condition that X be compact is necessary.

We will let X be the set of real numbers

$$X = \{n-n^{-1}: n = 1, 2, \dots\}$$

with metric

$$d(x, y) = |x-y|$$

for all x, y in X. X is obviously complete with this metric but not compact.

Define a mapping T of X into itself by

$$T\{n-n^{-1}) = n+1-(n+1)^{-1}$$

for $n=1, 2, \dots$. Then supposing that $m<n$, we have

$$d\big(T(m-m^{-1}), T(n-n^{-1})\big) = n-m-(n+1)^{-1}+(m+1)^{-1}$$

and

$$d\big(m-m^{-1}, T(n-n^{-1})\big)+d\big(n-n^{-1}, T(m-m^{-1})\big) =$$

$$= -m+m^{-1}+n+1-(n+1)^{-1}+n-n^{-1}-(m+1)+(m+1)^{-1} =$$

$$= 2[n-m-(n+1)^{-1}+(m+1)^{-1}]+[(n+1)^{-1}-n^{-1}+m^{-1}-(m+1)^{-1}].$$

Since $m<n$, it follows that

$$(n+1)^{-1}-n^{-1}+m^{-1}-(m+1)^{-1} > 0$$

and so

$$d\big(T(m-m^{-1}), T(n-n^{-1})\big) < \frac{1}{2}\left[d\big(m-m^{-1}, T(n-n^{-1})\big)+d\big(n-n^{-1}, T(m-m^{-1})\big)\right]$$

if $m<n$ and by the symmetry this inequality now of course holds if $m \neq n$.

T is obviously continuous and so this example satisfies all the hypothesese of the theorem with the exception of the compactness of X, but T has no fixed point.

We finally note that the condition that the inequality does not have to hold if $x=y$ is also essential, otherwise we would have

$$d(Tx, Tx) < d(x, Tx)$$

and no fixed point of T could satisfy this inequality.

Reference

[1] B. Fisher, A fixed point theorem, *Mathematics Magazine*, **48**, (1975), 223—225.

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