Semihereditary polynomial rings

Dedicated to the memory of Andor Kertész

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Recently, CAMILLO [2] and MCCARTHY [6] have shown that a commutative ring A is von Neumann regular if and only if the polynomial ring $A[x]$ is semihereditary. Here we offer a non-commutative extension of this result, with a proof by completely different methods.

Our first preliminary result completes by different methods a result of ENDO [3] and ENOCHS [4]. We first recall that BOURBAKI [1] calls a ring left coherent if all its finitely generated left ideals are finitely presented.

Proposition. *A ring is left semihereditary if and only if it is left coherent and has flat (=weak) global dimension at most one.*

PROOF. All of these properties can be tested by using finitely generated left ideals. Such an ideal is projective if and only if it is finitely presented and flat.

For any associative ring A with identity we can now show:

Theorem. *The polynomial ring $A[x]$ is left semihereditary if and only if $A[x]$ is left coherent and A is von Neumann regular.*

PROOF. Using the proposition we see that $A[x]$ is left semihereditary if and only if $A[x]$ is left coherent and has flat global dimension at most one. But JENSEN [5] has shown that the flat global dimension of $A[x]$ exceeds that of A by precisely one, from which the result is immediate.

Corollary. *If A is commutative, then $A[x]$ is semihereditary if and only if A is regular.*

PROOF. SABBAGH [7] has shown that if A is commutative von Neumann regular then $A[x]$ is coherent, from which the result is immediate.

References

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