On characterizations of certain classes of semigroups

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Several important classes of semigroups or associative rings have been characterized by ideal-theoretical relations (see [2], [3], [4], [5]). These characterizations are given by equalities. In this paper three important classes of semigroups will be characterized by ideal-theoretical inequalities. These classes consist of regular and/or intraregular semigroups.

For the notions and notations used in this paper we refer to A. H. CLIFFORD and G. B. PRESTON [1].

Theorem 1. *For a semigroup S the following conditions are pairwise equivalent:*
($\mathcal{A}$) *S is regular.*
($\mathcal{B}$) *$B \cap R \subseteqq RB$ for every bi-ideal B and every right ideal R of S.*
($\mathcal{C}$) *$B(a) \cap R(b) \subseteqq R(b)B(a)$ for every couple a, b of elements in S.*
($\mathcal{D}$) *$B(a) \cap R(a) \subseteqq R(a)B(a)$ for every element a of S.*

PROOF. ($\mathcal{A}$) implies ($\mathcal{B}$). Let S be a regular semigroup. Then any bi-ideal B of S can be written in the form

$$B = L' \cap R' = R'L', \tag{1}$$

where L' is a left ideal, R' is a right ideal of S (cf. [5], Theorem 29). Since one-sided ideals of S are evidently globally idempotent, we have (by making use of the Kovács—Iséki criterion [1], p. 34)

$$B \cap R = L' \cap R' \cap R \subseteqq L' \cap RR' = RR'L' = RB \tag{2}$$

for every bi-ideal B and every right ideal R of S.

The implications $(\mathcal{B}) \Rightarrow (\mathcal{C}) \Rightarrow (\mathcal{D})$ are trivial.

($\mathcal{D}$) implies ($\mathcal{A}$). Let S be a semigroup with property ($\mathcal{D}$). Then ($\mathcal{D}$) implies

$$L(a) \cap R(a) \subseteqq R(a)L(a) \tag{3}$$

for any element a of S. But $R(a)L(a) \subseteqq L(a) \cap R(a)$ also holds, thus we get

$$L(a) \cap R(a) = R(a)L(a) \tag{4}$$

for every element a of S. This means (see [3]) that every element a of S is regular.

REMARK 1. It is easy to see that Theorem 1 remains true with quasi-ideal instead of bi-ideal.

Theorem 2. *For a semigroup S the following conditions are equivalent:*
($\mathscr{A}$) *S is intraregular.*
($\mathscr{B}$) *$L\cap R\subseteqq LR$ for every left ideal L and every right ideal R of S.*
($\mathscr{C}$) *$L(a)\cap R(b)\subseteqq L(a)R(b)$ for every couple a, b of elements in S.*
($\mathscr{D}$) *$L(a)\cap R(a)\subseteqq L(a)R(a)$ for every element a of S.*

PROOF. ($\mathscr{A}$) implies ($\mathscr{B}$). Suppose that S is an intraregular semigroup, L is a left ideal, R is a right ideal of S, and $a\in L\cap R$. Then the product LR is a two-sided ideal of S, and $a^2\in LR$ implies $a\in LR$ because the ideals of S are semiprime (cf. [1], p. 121). Thus ($\mathscr{A}$) implies ($\mathscr{B}$) indeed.

Evidently $(\mathscr{B})\Rightarrow(\mathscr{C})\Rightarrow(\mathscr{D})$.

($\mathscr{D}$) implies ($\mathscr{A}$). Let S be a semigroup with property ($\mathscr{D}$). Then we have

$$a\in L(a)\cap R(a) \subseteqq L(a)R(a) \subseteqq S^1a^2S^1 \tag{5}$$

for every element a of S. Hence it follows easily that S is intraregular.

For an earlier ideal-theoretic characterization of intraregular semigroups, see G. SZÁSZ [7].

Theorem 3. *For a semigroup S the following conditions are equivalent:*
($\mathscr{A}$) *S is regular and intraregular.*
($\mathscr{B}$) *$A\cap B\subseteqq AB$ for every couple A, B of bi-ideals of S.*
($\mathscr{C}$) *$B\cap Q\subseteqq BQ$ for every bi-ideal B and every quasi-ideal Q of S.*
($\mathscr{D}$) *$P\cap Q\subseteqq PQ$ for every couple P, Q of quasi-ideals of S.*
($\mathscr{E}$) *$L\cap R\subseteqq LR\cap RL$ for every left ideal L and every right ideal R of S.*
($\mathscr{F}$) *$L(a)\cap R(b)\subseteqq L(a)R(b)\cap R(b)L(a)$ for every couple a, b of elements in S.*
($\mathscr{G}$) *$L(a)\cap R(a)\subseteqq L(a)R(a)\cap R(a)L(a)$ for every element a of S.*

PROOF. ($\mathscr{A}$) implies ($\mathscr{B}$). Let S be a semigroup which is both regular and intraregular. Then the bi-ideals of S are globally idempotent (see [5], Theorem 39). Hence it follows ($\mathscr{B}$), because

$$A\cap B = (A\cap B)^2 \subseteqq AB \tag{6}$$

holds for any two bi-ideals A, B of S.

Evidently $(\mathscr{B})\Rightarrow(\mathscr{C})\Rightarrow(\mathscr{D})\Rightarrow(\mathscr{E})\Rightarrow(\mathscr{F})\Rightarrow(\mathscr{G})$.

($\mathscr{G}$) implies ($\mathscr{A}$). Let S be a semigroup with property ($\mathscr{G}$). Then ($\mathscr{G}$) implies (3) resp. (4) for any element a of S. Thus S is regular. On the other hand, ($\mathscr{G}$) implies condition ($\mathscr{D}$) of our Theorem 2. Hence S is intraregular.

Theorem 3 is completely proved.

REMARK 2. The first author proved in [4] that a semigroup S is a semilattice of groups if and only if the condition

$$L\cap R = LR \tag{7}$$

holds for every left ideal L and every right ideal R of S. This criterion and our Theorem 2 imply that a semigroup which is a semilattice of groups is necessarily intraregular. Similarly it can be shown that also the semilattices of left [right] groups are intraregular semigroups.

For further characterizations by ideal-theoretical inequalities, see [6].

References

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