Periodic generalized functions

By Á. SZÁZ (Debrecen)

Introduction

In this paper, the theory of periodic distributions [1, 2, 3, 6, 7] is generalized and simplified. The development parallels that given in [4], where periodic convolution quotients were constructed to generalize periodic distributions.

According to the results of [10], we consider multipliers [5] from the convolution algebra of all trigonometric polynomials into itself instead of periodic distributions and periodic convolution quotients. This yields some simplifications in the corresponding theory and is in harmony with the conceptions of [8] and [9].

One of the most important features of the resulting theory is that it frees the theory of periodic distributions from certain difficulties that arise because not every trigonometric series converges there.

§ 1. The convolution algebra of continuous periodic functions

Let $C_{2\pi}=C_{2\pi}(\mathbf{R}^m)$ be the set of all continuous functions from $\mathbf{R}^m$ into $\mathbf{C}$ that are 2π-periodic in each variable. It is known that $C_{2\pi}$ with the usual linear operations, with the periodic convolution

$$(f*g)(x)=\frac{1}{(2\pi)^m}\int\limits_{[-\pi,\pi]^m} f(x-t)\,g(t)\,dt$$

and with the supremum norm

$$\|f\|=\sup_{x\in\mathbf{R}^m}|f(x)|$$

is a commutative Banach algebra with approximate identities.

For each $k\in\mathbf{Z}^m$, the function e_k defined on $\mathbf{R}^m$ by

$$e_k(x)=e^{ikx}=\exp i\sum_{j=1}^{m}k_jx_j$$

belongs to $C_{2\pi}$. Moreover, the set $T=T(\mathbf{R}^m)$ of all trigonometric polynomials, i.e., the linear hull of $\{e_k\}_{k\in\mathbf{Z}^m}$ in $C_{2\pi}$, is a dense ideal in $C_{2\pi}$.

For $f\in C_{2\pi}$, the numbers

$$\hat{f}(k)=(f*e_k)(0)=\frac{1}{(2\pi)^m}\int\limits_{[-\pi,\pi]^m} e^{-ikt}f(t)\,dt$$

are called the Fourier coefficients of f. It is known that the set $\mathbf{C}^{\mathbf{Z}^m}$ of all functions from $\mathbf{Z}^m$ into $\mathbf{C}$ equipped with the usual pointwise operations and with the topology induced by the family of seminorms

$$|A|_k = |A(k)| \quad (A \in \mathbf{C}^{\mathbf{Z}^m})$$

is a commutative Fréchet algebra with identity such that the mapping

$$f \to \hat{f}$$

is a continuous algebraic isomorphism of $C_{2\pi}$ into $\mathbf{C}^{\mathbf{Z}^m}$.

For $f \in C_{2\pi}$ and $\lambda \in \mathbf{R}^m$, the translate $\tau_\lambda f$ of f,

$$(\tau_\lambda f)(x) = f(x+\lambda),$$

belongs to $C_{2\pi}$ and we have

$$\tau_\lambda(f*g) = (\tau_\lambda f)*g \quad \text{and} \quad (\tau_\lambda f)\hat{\ }(k) = e^{ik\lambda}\hat{f}(k)$$

for all $g \in C_{2\pi}$ and $k \in \mathbf{Z}^m$.

Finally, if $f \in C_{2\pi}$ and $\lambda \in \mathbf{R}^m$ such that the directional derivative $\partial_\lambda f$ of f,

$$(\partial_\lambda f)(x) = \lim_{t \to 0} \frac{1}{t}\left(f(x+t\lambda) - f(x)\right),$$

exists and is continuous on $\mathbf{R}^m$, then it belongs to $C_{2\pi}$ and we have

$$\partial_\lambda(f*g) = (\partial_\lambda f)*g \quad \text{and} \quad (\partial_\lambda f)\hat{\ }(k) = ik\lambda\hat{f}(k)$$

for all $g \in C_{2\pi}$ and $k \in \mathbf{Z}^m$.

§ 2. The convolution algebra of periodic generalized functions

Definition 2.1. A function F from T into T is called a periodic generalized function if

$$F(\varphi)*\psi = \varphi * F(\psi)$$

for all $\varphi, \psi \in T$.

Example 2.2. If $f \in C_{2\pi}$, then the function F_f defined on T by

$$F_f(\varphi) = f * \varphi$$

is a periodic generalized function.

Proposition 2.3. *Let F be a periodic generalized function. Then F is linear and*

$$F(\varphi * \psi) = F(\varphi) * \psi$$

for all $\varphi \in T$ and $\psi \in C_{2\pi}$.

Proof. If $\varphi \in T$ and $\psi \in C_{2\pi}$, then

$$F(\varphi*\psi)*e_k = (\varphi*\psi)*F(e_k) = \left(\varphi*F(e_k)\right)*\psi = \left(F(\varphi)*e_k\right)*\psi =$$
$$= \left(F(\varphi)*\psi\right)*e_k,$$

and hence

$$\big(F(\varphi * \psi) - F(\varphi) * \psi\big)\hat{\ }(k) = \big((F(\varphi * \psi) - F(\varphi) * \psi) * e_k\big)(0) = 0$$

for all $k \in \mathbf{Z}^m$, i.e., $F(\varphi * \psi) - F(\varphi) * \psi = 0$.

A similar argument can be used to see that F is linear.

Definition 2.4. If F is a periodic generalized function, then the numbers

$$\hat{F}(k) = F(e_k)(0) \quad (k \in \mathbf{Z}^m)$$

are called the Fourier coefficients of F.

Proposition 2.5. *Let F be a periodic generalized function. Then*

$$F(\varphi) = \sum_{k \in \mathbf{Z}^m} \hat{F}(k) \hat{\varphi}(k) e_k$$

for all $\varphi \in T$.

PROOF. If $\varphi \in T$, then $\hat{\varphi}(k) \neq 0$ only for finitely many $k \in \mathbf{Z}^m$, and

$$\varphi = \sum_{k \in \mathbf{Z}^m} \hat{\varphi}(k) e_k.$$

Hence, since F is linear and

$$F(e_k) = F(e_k * e_k) = F(e_k) * e_k = \big(F(e_k) * e_k\big)(0) e_k = F(e_k)(0) e_k = \hat{F}(k) e_k$$

the assertion immediately follows.

Definition 2.6. Let $T^* = T^*(\mathbf{R}^m)$ be the set of all periodic generalized functions.

For $F, G \in T^*$ and $\alpha \in \mathbf{C}$ let $F+G$, αF and $F * G$ be the functions defined on T by

$$(F+G)(\varphi) = F(\varphi) + G(\varphi), \quad (\alpha F)(\varphi) = \alpha F(\varphi)$$

and

$$(F * G)(\varphi) = F\big(G(\varphi)\big).$$

Moreover, for $k \in \mathbf{Z}^m$, let $\| \ \|_k$ be the functional defined on T^* by

$$\|F\|_k = \|F(e_k)\|.$$

Theorem 2.7. *With the corresponding operations and with the topology induced by the family of seminorms $\| \ \|_k$, T^* is a commutative Fréchet algebra with identity such that the mapping defined on T^* by*

$$F \to \hat{F}$$

is an algebraic and topological isomorphism of T^* onto $\mathbf{C}^{\mathbf{Z}^m}$.

PROOF. By Proposition 2.5, it is clear that the mapping $F \to \hat{F}$ is injective. To prove that this mapping is into $\mathbf{C}^{\mathbf{Z}^m}$, suppose that $A \in \mathbf{C}^{\mathbf{Z}^m}$. Let F be the function defined on T by

$$F(\varphi) = \sum_{k \in \mathbf{Z}^m} A(k) \hat{\varphi}(k) e_k.$$

Then a simple computation shows that $F \in T^*$ and $\hat{F} = A$.

Straightforward calculations show that if $F, G \in T^*$ and $\alpha \in \mathbf{C}$, then $F+G$, αF, $F*G \in T^*$ and

$$(F+G)\hat{} = \hat{F}+\hat{G}, \quad (\alpha F)\hat{} = \alpha\hat{F}, \quad (F*G)\hat{} = \hat{F}\hat{G}.$$

For example,

$$(F*G)\hat{}\,(k) = (F*G)(e_k)(0) = F\big(G(e_k)\big)(0) = F\big(\hat{G}(k)e_k\big)(0) =$$
$$= \big(\hat{G}(k)F(e_k)\big)(0) = \big(\hat{F}(k)\hat{G}(k)e_k\big)(0) = \hat{F}(k)\hat{G}(k)$$

shows that $(F*G)\hat{} = \hat{F}\hat{G}$.

Finally, to complete the proof observe that

$$\|F\|_k = |\hat{F}|_k$$

for all $F \in T^*$ and $k \in \mathbf{Z}^m$.

Corollary 2.8. Let $F \in T^*$. Then the following propositions are pairwise equivalent:

(i) F is invertible in T^*;
(ii) F is not a divisor of zero in T^*;
(iii) $\hat{F}(k) \neq 0$ for all $k \in \mathbf{Z}^m$.

Moreover, if F is invertible in T^*, then F is injective, the inverse function F^{-1} of F is the inverse of F in T^* and

$$F^{-1}(\varphi) = \sum_{k \in \mathbf{Z}^m} \frac{1}{\hat{F}(k)} \hat{\varphi}(k) e_k$$

for all $\varphi \in T$.

PROOF. This follows immediately from Theorem 2.7 and Proposition 2.5.

Corollary 2.9. *Let (F_α) be a net in T^* and $F \in T^*$. Then the following propositions are pairwise equivalent:*

(i) $\lim_\alpha F_\alpha = F$ *in* T^*;

(ii) $\lim_\alpha \hat{F}_\alpha(k) = \hat{F}(k)$ *for all* $k \in \mathbf{Z}^m$;

(iii) $\lim_\alpha F_\alpha(\varphi) = F(\varphi)$ *in* $C_{2\pi}$ *for all* $\varphi \in T$.

PROOF. It is clear, that (i) and (ii) are equivalent, and that (iii) implies (i).

To prove that (ii) implies (iii), suppose that $\varphi \in T$. Then there exists $n \in \mathbf{N}$, such that $\hat{\varphi}(k) = 0$ if $k \in \mathbf{Z}^m$ and $|k| > n$, where $|k| = \sum_{j=1}^{m} |k_j|$. Thus, by Proposition 2.5, we have

$$\|F_\alpha(\varphi) - F(\varphi)\| = \Big\| \sum_{|k| \leqq n} \big(\hat{F}_\alpha(k) - \hat{F}(k)\big)\hat{\varphi}(k)e_k \Big\| \leqq \sum_{|k| \leqq n} |\hat{F}_\alpha(k) - \hat{F}(k)|\,|\hat{\varphi}(k)| \leqq$$
$$\leqq \max_{|k| \leqq n} |\hat{F}_\alpha(k) - \hat{F}(k)| \sum_{|k| \leqq n} |\hat{\varphi}(k)|$$

for all α. Hence, it is clear that $\lim_\alpha F_\alpha(\varphi) = F(\varphi)$ in $C_{2\pi}$.

§ 3. Embedding of complex numbers and continuous periodic functions

Definition 3.1. For $\alpha \in \mathbf{C}$ and $f \in C_{2\pi}$ let F_α and F_f be the functions defined on T by

$$F(\varphi) = \alpha\varphi \quad \text{and} \quad F_f(\varphi) = f * \varphi.$$

Theorem 3.2. (i) *The mapping defined on* $\mathbf{C}$ *by*

$$\alpha \to F_\alpha$$

is an algebraic and topological isomorphism of $\mathbf{C}$ *into* T^*.

(ii) *The mapping defined on* $C_{2\pi}$ *by*

$$f \to F_f$$

is a continuous algebraic isomorphism of $C_{2\pi}$ *into* T^* *such that* $\hat{F}_f = \hat{f}$ *for all* $f \in C_{2\pi}$.

(iii) *For* $\alpha \in \mathbf{C}$ *and* $f \in C_{2\pi}$, *we have* $F_\alpha = F_f$ *if and only if* $\alpha = 0$ *and* $f = 0$.

Proof. To prove (i) and (ii) is left to the reader. We prove only the nontrivial part of (iii). For this, suppose that $\alpha \in \mathbf{C}$ and $f \in C_{2\pi}$ such that $F_\alpha = F_f$. Then

$$\alpha = \hat{F}_\alpha(k) = \hat{F}_f(k) = \hat{f}(k)$$

for all $k \in \mathbf{Z}^m$. Hence, by the Riemann—Lebesgue lemma,

$$\alpha = \lim_{|k| \to \infty} \hat{f}(k) = 0.$$

Thus $\hat{f}(k) = 0$ for all $k \in \mathbf{Z}^m$. This implies that $f = 0$.

Definition 3.3. For $\alpha \in \mathbf{C}$ and $f \in C_{2\pi}$ identify α with F_α and f with F_f by writing

$$\alpha = F_\alpha \quad \text{and} \quad f = F_f.$$

Corollary 3.4. *Let* $F \in T^*$. *Then*

$$F * \varphi = F(\varphi)$$

for all $\varphi \in T$.

Proof. If $\varphi \in T$, then we have

$$(F * \varphi)\hat{}\,(k) = \hat{F}(k)\hat{\varphi}(k) = F(\varphi)\hat{}\,(k)$$

for all $k \in \mathbf{Z}^m$. This implies that $F * \varphi = F(\varphi)$.

Corollary 3.5. *Let* $F \in T^*$. *Then*

$$F = \sum_{k \in \mathbf{Z}^m} \hat{F}(k) e_k$$

in T^*.

Proof. For each $n \in \mathbf{N}$, let

$$F_n = \sum_{|k| \leqq n} \hat{F}(k) e_k.$$

Then $\hat{F}_n(k) = \hat{F}(k)$ if $k \in \mathbf{Z}^m$ and $|k| \leqq n \in \mathbf{N}$. Thus

$$\lim_{n \to \infty} \hat{F}_n(k) = \hat{F}(k)$$

for all $k\in\mathbf{Z}^m$. Hence, by Corollary 2.9, it follows that

$$\lim_{n\to\infty} F_n = F$$

in T^*.

Corollary 3.6. *T is a dense ideal in T^*.*

PROOF. This follows immediately from Corollaries 3.4 and 3.5.

§ 4. Translation and differentiation

Definition 4.1. For $\lambda\in\mathbf{R}^m$, let T_λ be the function defined on T by

$$T_\lambda(\varphi) = \tau_\lambda\varphi.$$

Theorem 4.2. *Let $\lambda\in\mathbf{R}^m$. Then*
(i) $T_\lambda\in T^*$;
(ii) $\hat{T}_\lambda(k)=e^{ik\lambda}$ *for all* $k\in\mathbf{Z}^m$;
(iii) $T_\lambda * f=\tau_\lambda f$ *for all* $f\in C_{2\pi}$.

PROOF. (i) and (ii) are obvious, namely T_λ is linear and $T_\lambda(e_k)=e^{ik\lambda}e_k$ for all $k\in\mathbf{Z}^m$.

If $f\in C_{2\pi}$, then we have

$$(T_\lambda * f)^\wedge(k) = \hat{T}_\lambda(k)\hat{f}(k) = {}^{ik\lambda}\hat{f}(k) = (\tau_\lambda f)^\wedge(k)$$

for all $k\in\mathbf{Z}^m$. This implies (iii).

Definition 4.3. For $\lambda\in\mathbf{R}^m$, let D_λ be the function defined on T by

$$D_\lambda(\varphi) = \partial_\lambda\varphi.$$

Theorem 4.4. *Let $\lambda\in\mathbf{R}^m$. Then*
(i) $D_\lambda\in T^*$;
(ii) $\hat{D}_\lambda(k)=ik\lambda$ *for all* $k\in\mathbf{Z}^m$;
(iii) $D_\lambda * f=\partial_\lambda f$ *if* $f\in C_{2\pi}$ *such that* $\partial_\lambda f\in C_{2\pi}$.

PROOF. (i) and (ii) are obvious, namely D_λ is linear and $D_\lambda(e_k)=ik\lambda e_k$ for all $k\in\mathbf{Z}^m$.

If $f\in C_{2\pi}$ such that $\partial_\lambda f\in C_{2\pi}$, then we have

$$(D_\lambda * f)^\wedge(k) = \hat{D}_\lambda(k)\hat{f}(k) = ik\lambda\hat{f}(k) = (\tau_\lambda f)^\wedge(k)$$

for all $k\in\mathbf{Z}$. This implies (iii).

Remark 4.5. If $u_j\in\mathbf{R}^m$ such that the jth coordinate of u_j is 1 and the others are 0, then we write $D_j=D_{u_j}$. It is clear that

$$D_\lambda = \sum_{j=1}^m \lambda_j D_j$$

for all $\lambda\in\mathbf{R}^m$.

Theorem 4.6. *Let* $\lambda \in \mathbf{R}^m$. *Then*

$$\lim_{t\to 0} \frac{1}{t}(T_{t\lambda}-1) = D_\lambda$$

in T^*.

PROOF. Let $(t_n)_{n=0}^\infty$ be a null sequence in $\mathbf{R}$ such that $t_n \neq 0$ for all $n \in \mathbf{N}$. Moreover, for $n \in \mathbf{N}$, let

$$F_n = \frac{1}{t_n}(T_{t_n\lambda}-1).$$

Then

$$\lim_{n\to\infty} \hat{F}_n(k) = \lim_{n\to\infty} \frac{1}{t_n}(e^{ikt_n\lambda}-1) = ik\lambda = \hat{D}_\lambda(k)$$

for all $k \in \mathbf{Z}^m$. By Corollary 2.9, this implies that

$$\lim_{n\to\infty} F_n = D_\lambda$$

in T^*.

Remark 4.7. Having been defined the exponential function of periodic generalized functions it can be shown that

$$T_\lambda = e^{D_\lambda}$$

for all $\lambda \in \mathbf{R}^m$.

Note. A preliminary version of this paper was delivered at the Conference on Generalized Functions and Operational Calculus, Varna (Bulgaria), September 29—October 6, 1975.

Finally, we remark that a similar theory can be obtained by considering linear functionals on T, but in this case the convolution becomes more difficult.

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(Received November 3, 1975.)