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On generalizations of Ingham–Jessen's and Mikolás' inequalities

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Abstract. A generalization of the well-known Ingham–Jessen's inequality is given in terms of certain functionals. It is then used to prove a generalization of Mikolás' inequality and its refinement.

In this paper we shall generalize the well-known Ingham–Jessen's inequality (Theorem 1), and then using this result we shall give a generalization of the Mikolás' inequality (Theorem 2).

Let us point out that the Ingham–Jessen's inequality was proved in [2], where also a type of generalization of this inequality was given. Yet another types of generalizations of this inequality are presented in [3], esp. p. 246. A simple proof of the Mikolás' inequality was shown in [5] and we shall prove its generalization in terms of certain functionals.

The main ingredients of our proofs are some results from [1] about generalized power means, which enable us to give a different (new) proof of our Theorem 1, unlike the one given in [4]. In fact, we will only need generalized Hölder's and Minkowski's inequalities and the monotone property of the generalized power means.

Let us first recall some definitions and facts from [1]. Let S be a nonempty set and L a class of mappings $f : S \longrightarrow \mathbb{R}_+$ from S into the non-negative reals $\mathbb{R}_+$. We shall consider functionals $A : L \longrightarrow \mathbb{R}_+$ with the following properties:

a) $f \in L, \lambda > 0 \Longrightarrow \lambda f \in L$ and $A(\lambda f) = \lambda A(f)$,
b) $1 \in L$ and $A(1) = 1$,
c) $f, g \in L,\ f \le g$ (i.e. $f(t) \le g(t),\ \forall t \in S) \Longrightarrow A(f) \le A(g)$,
d) $f, g \in L \Longrightarrow f + g \in L$ and $A(f+g) \le A(f) + A(g)$.

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(They are called *positive homogeneity, normality, monotonicity* and *sub-additivity*, respectively.)

Throughout this paper we shall always assume that all expressions are well defined. So, for example, whenever we write $A(fg)$ or $A(f^p)$, we shall assume that $fg \in L$ or $f^p \in L$.

For any nonzero real number p, define the *generalized power mean* of f by:

$$M_p(f) = A(f^p)^{1/p}.$$

Then the following was proved (Theorems 5–8) in [1], assuming that all functions $f, g, g_1, \dots, g_n \in L$ and $A : L \longrightarrow \mathbb{R}_+$ satisfies a) – d):

1) If $k_1, k_2, \dots, k_n$ are positive numbers with $\sum_{i=1}^{n} 1/k_i = 1/p$, then

$$M_p\left(\prod_{i=1}^{n} g_i\right) \leq \prod_{i=1}^{n} M_{k_i}(g_i) \text{ (generalized Hölder's inequality).}$$

2) If $p > 0$, $k_1 > 0$, $k_2 < 0, \dots, k_n < 0$ with $\sum_{i=1}^{n} 1/k_i = 1/p$, then the reverse of the above Hölder's inequality holds.

3) If $p, q \neq 0$ and $p \leq q$ are real numbers, then

$M_p(f) \leq M_q(f)$ *(monotone property of power means).*

4) If $p > 1$ is any real number, then

$M_p(f + g) \leq M_p(f) + M_p(g)$ *(generalized Minkowski's inequality).*

Now we shall use these results (in fact only 1) and 4)) to prove the following result (Theorem 2 in [4]).

Theorem 1. *For any $r \leq s$, $rs > 0$, the following inequality holds*

$$(*) \qquad A\left\{M_n^{[r]}(\underline{f}; w)^s\right\}^{1/s} \leq M_n^{[r]}\left\{A(\underline{f}^s)^{1/s}; w\right\}.$$

Here A is any functional satisfying a)–d) and for given functions $f_1, \dots, f_n \in L$, we put $\underline{f}(t) = \{f_1(t), \dots, f_n(t)\}$ and we define

$$A(\underline{f}^s)^{1/s} = \left\{A(f_1^s)^{1/s}, \dots, A(f_n^s)^{1/s}\right\}.$$

Furthermore, for a positive sequence $\underline{a} = (a_1, \dots, a_n)$ with (positive) *weights* $w = (w_1, \dots, w_n)$ with $\sum_{i=1}^{n} w_i = 1$, the *power mean of order* r of $\underline{a}$ *with weight* w is defined by

$$M_n^{[r]}(\underline{a}; w) = \left(\sum_{i=1}^{n} w_i a_i^r\right)^{1/r}, \quad \text{if } r \neq 0,$$

$$= \prod_{i=1}^{n} a_i^{w_i}, \qquad \text{if } r = 0.$$

Proof. First of all, for $r = 0$, $s = 1$, the inequality $(*)$ follows immediately from inequality 1) by taking $g_i = f_i^{w_i}$, $k_i = 1/w_i$, $i = 1, \dots, n$, and $p = 1$. In this case, $(*)$ is simply

$$A\left(\prod_{i=1}^{n} f_i^{w_i}\right) \leq \prod_{i=1}^{n} A(f_i)^{w_i}.$$

To prove $(*)$ in the case $r = 0$ and $s > 0$ any number, we take this time $g_i = f_i^{w_i s}$, $k_i = 1/w_i$, $i = 1, \dots, n$ and $p = 1$ in 1).

Now we consider the case $r \neq 0$, $s \neq 0$, $r < s$. The generalized Minkowski's inequality 4) for any $p > 1$, reads as follows

$$A((f+g)^p)^{1/p} \leq A(f^p)^{1/p} + A(g^p)^{1/p}.$$

By induction on n, it follows at once from here that for any positive functions $g_1, \dots, g_n$ we have

$$A\left(\left(\sum_{i=1}^{n} g_i\right)^p\right)^{1/p} \leq \sum_{i=1}^{n} A(g_i^p)^{1/p}.$$

By taking here $g_i = w_i F_i$, with $\sum_{i=1}^{n} w_i = 1$, we get

$$A\left(\left(\sum_{i=1}^{n} w_i F_i\right)^p\right)^{1/p} \leq \sum_{i=1}^{n} w_i A(F_i^p)^{1/p}.$$

If $0 < r < s$, set $p = s/r$ and $F_i = f_i^r$. Then we get from here

$$A\left(\left(\sum_{i=1}^{n} w_i f_i^r\right)^{s/r}\right)^{r/s} \leq \sum_{i=1}^{n} w_i A(f_i^s)^{r/s}.$$

By raising both sides to the power $1/r$, we obtain exactely $(*)$. Similarly, for $r < s < 0$, set $p = r/s$ and $F_i = f_i^s$, and then raising both sides to the power $1/s$ gives again $(*)$. $\square$

Now using Theorem 1 and monotone property of power means (i.e. inequality 3), we shall prove a generalization of Mikolás' inequality and its refinement as given by Alzer in [5], for functionals.

Theorem 2. *Let $A : L \longrightarrow \mathbb{R}_+$ be any functional satisfying conditions a), c) and d) with $A(1) \neq 0$ and let p, r, t be positive numbers such that $rp \leq t, r \leq t$. Then for any $\underline{f} = (f_1, \dots, f_n), f_1, \dots, f_n \in L$, and weight $w = (w_1, \dots, w_n)$ we have*

$$A\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^p\right\} \leq A(1)^{1-rp/t} A\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^{t/r}\right\}^{rp/t}$$

$$\leq A(1)^{1-rp/t}\left(\sum_{j=1}^{n} w_j A(f_j^t)^{r/t}\right)^p.$$

Proof. Let A' be a functional satisfying conditions a)–d). Then by monotone property 3) and Theorem 1 we have

$$A'\{M_n^{[r]}(\underline{f}; w)^{rp}\}^{1/rp} \leq A'\{M_n^{[r]}(\underline{f}; w)^t\}^{1/t}$$

$$\leq M_n^{[r]}\{A'(\underline{f}^t)^{1/t}; w\}.$$

By definition of $M_n^{[r]}$, this implies

$$A'\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^{1/r \cdot rp}\right\}^{1/rp} \leq A'\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^{t/r}\right\}^{1/t}$$

$$\leq \left\{\sum_{j=1}^{n} w_j A'(f_j^t)^{r/t}\right\}^{1/r}.$$

By raising this to the power rp we get

$$A'\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^p\right\} \leq A'\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^{t/r}\right\}^{rp/t}$$
$$\leq \left\{\sum_{j=1}^{n} w_j A'(f_j^t)^{r/t}\right\}^p.$$

Now we take A' defined by $A'(f) = A(f)/A(1)$. Then these inequalities yield

$$\frac{1}{A(1)} A\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^p\right\} \leq \frac{1}{A(1)^{rp/t}} A\left\{\left(\sum_{j=1}^{n} w_j f_j^r\right)^{t/r}\right\}^{rp/t}$$
$$\leq \frac{1}{A(1)^{rp/t}} \left(\sum_{j=1}^{n} w_j A(f_j^t)^{r/t}\right)^p.$$

By multiplying with $A(1)$, we obtain Theorem 2. □

Corollary (Mikolás' inequality)**.** *Let* $x_{ij} \geq 0$, $i = 1, \dots, m$, $j = 1, \dots, n$, *and let* p, r, t *be positive numbers such that* $rp \leq t$ *and* $r \leq t$. *Then*

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}^r\right)^p \leq m^{1-rp/t}\left[\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}^r\right)^{t/r}\right]^{rp/t}$$
$$\leq m^{1-rp/t}\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}^t\right)^{r/t}\right]^p.$$

(In fact, only the case $t = 1$ was considered in [5].)

Proof. Let $S = \{x = (x_{11}, \dots, x_{1n}, \dots, x_{m1}, \dots, x_{mn}) \in \mathbb{R}^{mn} \mid x_{ij} \geq 0, \forall i, j\}$ and for any function $f : S \longrightarrow \mathbb{R}_+$, let $A(f) = \sum_{i=1}^{m} A_i(f)$, where A_i's satisfy a)–d). Then A satisfies a), c) and d) and $A(1) = m$. Now let $f_1, \dots, f_n : S \longrightarrow \mathbb{R}_+$ be given functions and $w_1 = \dots = w_n (= 1/n)$.

By putting $A_i(f_j)(x) = x_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$, the claim follows immediately from Theorem 2. $\square$

As it was noticed in [5] for $t = 1$, if $r \geq t$ and $rp \geq t$, the converse of the above inequalities hold and the case of equality can be easily obtained.

As a final remark, note that if we take in Theorem 2 for the functional A to be a certain integral, and/or if summations we transform into integrals, we will obtain some interesting integral inequalities, but we shall not consider it here.

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