Standard ideals in matroid lattices

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1. Introduction

In the papers [6], [7] and [8] for some classes of *AC*-lattices L necessary and sufficient conditions were given for $F(L)$ (the ideal of the finite elements of L) to be a standard ideal.

If the underlying *AC*-lattice L is upper-continuous, that is, a matroid lattice, then one can assign to each element $a \in L$ a uniquely determined cardinal number $r(a)$ which will be called the rank of the element $a \in L$ [1]). The set

$$F_{\aleph}(L) = \{a \mid a \in L \text{ and } r(a) < \aleph\}$$

forms an ideal in L. We ask, under which conditions this ideal is standard. We answer this question by generalizing our above mentioned results concerning $F(L)$ to the ideal $F_{\aleph}(L)$.

The roots for our topic can be found in the paper [3] by M. F. JANOWITZ. We thank Dr. M. F. Janowitz also for his valuable remarks during our correspondence.

2. Basic notions

For two elements a, b of a lattice L, $(a, b)M$ means that the implication

$$c \leqq b \Rightarrow (c \vee a) \wedge b = c \vee (a \wedge b)$$

is true; if this is the case, we say that (a, b) is a modular pair. If the above implication is not true for the pair (a, b), then we write $(a, b)\overline{M}$.

If the implication

$$(a, b)M \Rightarrow (b, a)M$$

holds in a lattice L, then L is said to be an M-symmetric lattice.

A lattice L with 0 is called weakly modular, if in L the implication

$$a \wedge b \neq 0 \Rightarrow (a, b)M$$

holds.[2])

[1]) cf. also S. MACLANE, *A* lattice formulation for transcendence degrees and *p*-bases, *Duke Math. J.* **4** (1938), 455—468.

[2]) We remark that in [1] the notion „weakly modular" has been used in another sense.

Let now Λ be a given complemented modular lattice with the lattice operations $\sqcup$ and $\sqcap$ and let S denote a fixed subset of $\Lambda-\{0, 1\}$ with the properties

$$a \in S \quad \text{and} \quad 0 < b \leqq a \Rightarrow b \in S$$

and

$$a, b \in S \Rightarrow a \sqcup b \in S.$$

If we endow the set $L \equiv \Lambda - S$ with the order relation of Λ, then L becomes a weakly modular M-symmetric lattice (cf. [5, Theorem 3.11, p. 12]). If a weakly modular M-symmetric lattice L arises in the above described manner from a complemented modular lattice, then L is called a *Wilcox lattice*.

The elements of S are said to be the imaginary elements for L. If S has a greatest element i, then i is called the imaginary unit for L.

An ideal R of a lattice L is called standard, if $J \wedge (R \vee K) = (J \wedge R) \vee (J \wedge K)$ for every pair of ideals J, K of L (cf. [2]).

Let L be a lattice with 0. We say that $a \in L$ and $b \in L$ are perspective and write $a \sim b$, if there exist an element $x \in L$ such that

$$a \vee x = b \vee x \quad \text{and} \quad a \wedge x = b \wedge x = 0.$$

An ideal R of a lattice L with 0 is called p-ideal if $a \in R$ and $b \sim a$ implies $b \in R$.

In a lattice L we write $b \prec a$ $(a, b \in L)$ if $b < a$ and if $b \leqq x \leqq a$ implies either $x = b$ or $x = a$. Let L be a lattice with 0 and let $a, b \in L$. We write $a <| b$ if

$$a \wedge b = 0 \quad \text{and} \quad b \prec a \vee b.$$

If simultaneously $a <| b$ and $b <| a$ hold, then we call the elements a, b parallel and write $a \parallel b$ (cf. [5, Definition 17.1, p. 72]).

The following assertion is mentioned without proof.

Proposition 2.1. *Let L be a lattice with* 0. *Then every standard ideal of L is also a p-ideal of L.*

If $0 \prec p$ holds in a lattice with 0, then p is called an atom. A lattice L is said to be atomistic, if every element of L is a union of atoms. The covering property is defined as follows: if p is an atom and $p \not\leqq a$ $(a, p) \in L)$, then $a \prec a \vee p$. An atomistic lattice with covering property is called an AC-lattice. A matroid lattice is an upper-continuous AC-lattice.

3. Matroid lattices

We need the following

Proposition 3.1. *Let L be a matroid lattice and let $a, c \in L$ with $a \wedge c = 0$. Then there exists a maximal $m \geqq a$ for which $m \wedge c = 0$. Moreover, if $p \in L$ is an atom with $p \not\leqq m$, then $c \wedge (m \vee p) > 0$ and $c \wedge (m \vee p) <| m$.*

Proof. Consider the set

$$Y = \{y \mid y \in L,\ y \geqq a \text{ and } y \wedge c = 0\}.$$

$(Y, \leqq)$ is a partially ordered set (where $\leqq$ denotes the partial ordering of L). $Y \neq \emptyset$ since $a \in Y$. Let now $K = \{y_\nu \mid \nu \in \Gamma\}$ be a chain in Y. If we had

$$c \wedge \bigvee (y_\nu \mid \nu \in \Gamma) > 0$$

then there would exist an atom $q \in L$ such that

(1) $$q \leqq c$$

and

$$q \leqq \bigvee y_\nu \mid \nu \in \Gamma).$$

Since L is a matroid lattice, there exists a finite set $\{1, \ldots, n\}$ such that

(2) $$q \leqq q_{\nu_1} \vee \ldots \vee q_{\nu_n} \quad (\nu_i \in \Gamma;\ i = 1, \ldots, n),$$

where the q_{ν_i} are atoms and every q_{ν_i} is less or equal to a certain y_ν. By y^* we denote the greatest element among these (finitely many) y_ν. Then $q_{\nu_i} \leqq y^*$ $(1 \leqq i \leqq n)$. Because of (2), we get from this also

(3) $$q \leqq y^*.$$

(1) and (3) together yield

$$0 < q \leqq c \wedge y^*$$

which is a contradiction since $y^* \in Y$.

Hence $c \wedge \bigvee (y_\nu \mid \nu \in \Gamma) = 0$ which implies $\bigvee (y_\nu \mid \nu \in \Gamma) \in K$. This means that the chain K has an upper bound in Y. According to the Lemma of Kuratowski—Zorn Y has then a maximal element which we denote by m.

Let now $p \in L$ be an atom with $p \not\leqq m$. Then $m \prec m \vee p$ since L is an AC-lattice. Because of the maximality of m we have $m \vee p \notin Y$ and thus

(4) $$c \wedge (m \vee p) > 0.$$

Furthermore, we have

(5) $$c \wedge (m \vee p) \wedge m = c \wedge m = 0.$$

Moreover $c \wedge (m \vee p) \not\leqq m$, since $c \wedge (m \vee p) \leqq m$ implies $c \wedge (m \vee p) = c \wedge (m \vee p) \wedge m = c \wedge m = 0$ which is a contradiction to (4). Hence it follows that

(6) $$m \prec m \vee p = m \vee [c \wedge (m \vee p)].$$

(5) and (6) together mean that

$$c \wedge (m \vee p) <| \, m$$

and the proposition is proved.

Now we assign to every element of a matroid lattice a well-defined rank.

Proposition 3.2. *Let L be a matroid lattice. To every $(0 \neq)$ $x \in L$ there exists a uniquely determined cardinal number $r(x)$ which we call the rank of x.*

PROOF. Consider the set

$$A = A(x) = \{p \mid p \text{ atom and } p \leqq x\}$$

of all those atoms p of L which lie under x. For the atoms $p \in A$ and the subsets $P \subseteqq A$ we define a binary relation

$$D[p, A] \overset{\text{def}}{\Longleftrightarrow} p \leqq \bigvee P. \quad (7)$$ [3])

It is easy to see that the such defined binary relation satisfies the conditions (I)—(IV) of the abstract dependence in [4]. Thus (7) defines in A a dependence relation. Consider now in A two arbitrary maximal independent sets of atoms S and T. By [4, Corollary] we have $|S|=|T|$ [4]), that is, two maximal independent sets of atoms in A have the same power. This power, which is for every $(0 \neq)\ x \in L$ uniquely determined, will be called the rank $r(x)$ of x.
We supplement the rank notion by defining $r(0)=0$. The cardinal number $r(1)$ (1 denotes the greatest element of the matroid lattice) will be called the rank of L and will be denoted by $r(L)$.

Let now $a<b$ in a matroid lattice. By [5, Remark 13.2, p. 56] the interval $[a, b]$ is itself a matroid lattice. By $r[a, b]$ we denote the rank of $[a, b]$, that is, the rank of b with respect to the interval $[a, b]$. We put $r[a, a]=0$. Evidently $r[0, 1]=r(L)$.

A matroid lattice L is said to be of infinite length, if $r(L) \geqq \aleph_0$.

Proposition 3.3. *Let L be a matroid lattice with $a, b \in L$. Then*

$$r(a \vee b) \leqq r(a)+r(b).$$

Proof. Let P be a maximal independent set of atoms in $a \in L$ and Q a maximal independent set of atoms in $b \in L$. Hence

$$|P| = r(a) \quad \text{and} \quad |Q| = r(b).$$

Consider the set theoretic union $P \cup Q$. With the aid of the atoms contained in this union, the element $a \vee b$ can be represented: if $P=\{p_\alpha\}$ and $Q=\{q_\beta\}$, then $\bigvee p_\alpha \vee \bigvee q_\beta = a \vee b$. Consider now the sets P and $Q-P$. Evidently

$$P \cup (Q-P) = P \cup Q \quad \text{and} \quad P \cap (Q-P) = \emptyset$$

and therefore

$$r(a \vee b) = |P \cup (Q-P)| = |P|+|Q-P| =$$
$$= r(a)+|Q-P| \leqq r(a)+r(b).$$

Corollary 3.4. *Let L be a matroid lattice of infinite length. For every cardinal number $\aleph$ with $\aleph_0 \leqq \aleph \leqq r(L)$ we define*

$$F_\aleph(L) = \{a \mid a \in L \text{ and } r(a) < \aleph\}.$$

Then $F_\aleph(L)$ is ab ideal in L.

Proof. Let $a, b \in F_\aleph(L)$. Then $r(a), r(b) < \aleph$ and by Proposition 3.3. we have $r(a \vee b) \leqq r(a)+r(b)$ and thus $a \vee b \in F_\aleph(L)$. Let now $a \in F_\aleph(L)$ and $b \leqq a$. Then $r(b) \leqq r(a) < \aleph$ and therefore $b \in F_\aleph(L)$.

[3]) By $\bigvee P$ we denote the union of all atoms contained in P.
[4]) $|S|$ denotes the cardinal number of S.

For matroid lattices L, the ideal $F_{\aleph_0}(L)$ coincides with the set $F(L)$ as defined in [5, Definition 8.1, p. 35], that is, $F_{\aleph_0}(L)=F(L)$. $F(L)$ consists of the element 0 and of all those elements of L which can be represented as a union of finitely many atoms. $F(L)$ is also called the ideal of the finite elements of L; it is defined for arbitrary AC-lattices.

As a further preparation we need a theorem which characterizes the standard ideals in arbitrary lattices.

Proposition 3.5. (cf. [2, Theorem 2, p. 30]). *An ideal A of a lattice L is standard if and only if*

$$A\vee(x] = \{a\vee x_1 | a\in A \text{ and } x_1 \leqq x\}$$

holds for every principal ideal $(x]$ of L.

Now we are ready to generalize [7, Theorem 3.2] in the case of matroid lattices on arbitrary $F_{\aleph}(L)$.

Theorem 3.6. *Let L be a matroid lattice of infinite length and let $\aleph_0\leqq\aleph\leqq r(L)$. Then the following three conditions are equivalent:*

(i) *$F_{\aleph}(L)$ is a standard ideal in L;*

(ii) *if $[x, b\vee x]$ and $[b\wedge x, b]$ are transposed intervals and $r[x, b\vee x]<\aleph$ then also $r[b\wedge x, b]<\aleph$;*

(iii) *for the tripel $a, b, x\in L$ the following implication holds: if $b\leqq a\vee x$ and $a\in F_{\aleph}(L)$, then there exists an $a_1\in F_{\aleph}(L)$ such that $b=(b\wedge x)\vee a_1$.*

Proof. (i)$\Rightarrow$(ii): let $F_{\aleph}(L)$ be a standard ideal and let $r[x, b\vee x]<\aleph$. It follows that there exists in $[x, b\vee x]$ a set C of elements $\{c_\nu | c_\nu \succ x;\ \nu\in\Gamma\}$ such that $b\vee x = \bigvee (c_\nu | \nu\in\Gamma)$ and $|C|=|\{c_\nu | \nu\in\Gamma\}|<\aleph$. By [5, Lemma 8.18, p. 39] there exists to every c_ν an atom p_ν of L with the property

(8) $$c_\nu = x\vee p_\nu \quad (\text{for all } \nu\in\Gamma).$$

Consider the set $P=\{p_\nu\}$ of all those atoms of L which are by (8) assigned to the c_ν. Because of (8) we have $|P|\leqq|C|$ and hence $|P|<\aleph$. Put now

$$a \overset{\text{def}}{=} \bigvee (p_\nu | \nu\in\Gamma).$$

Then

$$x \leqq a\vee x = (\bigvee p_\nu)\vee x = \bigvee (p_\nu\vee x) = \bigvee c_\nu = b\vee x$$

and $r(a)<\aleph$; hence $a\in F_{\aleph}(L)$. From this it follows that $b\in F_{\aleph}(L)\vee(x]$. Thus by Proposition 3.5 there exists an

(9) $$x_1 \leqq x$$

and an

(10) $$a_1\in F_{\aleph}(L)$$

such that

$$b = x_1\vee a_1.$$

Because of (10) we have $r(a_1)<\aleph$. Hence we obtain (again using [5, Lemma 8.18, p. 39])

(11) $$r[x_1, x_1\vee a_1] = r[x_1, b] < \aleph.$$

Moreover we get by (9) and by $b=x_1\vee a_1$ the relation

$$x_1 \leqq x\wedge b \leqq b.$$

Using (11) we get from this

$$r[x\wedge b, b] < \aleph$$

which was to be proved.

(ii)⇒(iii): let $b\leqq x\vee a$ and $a\in F_\aleph(L)$. Then $r[x, x\vee a]<\aleph$. Because of $x\leqq x\vee b\leqq x\vee a$ we also have $r[x, x\vee b]<\aleph$. Then $r[x\wedge b, b]<\aleph$ follows by (ii). Similarly as in the proof of the implication (i)⇒(ii) one can show the existence of an $a_1\in F_\aleph(L)$ with $b=(x\wedge b)\vee a_1$.

(iii)⇒(i): this implication follows immediately from Proposition 3.5, and the theorem is proved.

It is not difficult to prove also [7, Lemma 3.4] in the case of matroid lattices for arbitrary $F_\aleph(L)$:

Proposition 3.7. *Let L be a matroid lattice of infinite length and let* $\aleph_0\leqq\aleph\leqq r(L)$. *Consider the following four conditions:*

(i) $F_\aleph(L)$ *is a standard ideal in L;*

(ii) $F_\aleph(L)$ *is a p-ideal of L;*

(iii) $y<|z\Rightarrow y\in F_\aleph(L)$;

(iv) $y\parallel z\Rightarrow y\in F_\aleph(L)$.

Then (i)⇒(ii)⇒(iii)⇒(iv).

Proof. (i)⇒(ii): this follows from Proposition 2.1.

(ii)⇒(iii): assume that there are elements $y, z\in L$ such that $y<|z$ but $y\notin F_\aleph(L)$. For an atom $p<y$ we obtain then $p\sim y$, which means that $F_\aleph(L)$ is not a *p*-ideal.

(iii)⇒(iv): this implication follows from the definition of parallelity.

4. Weakly modular matroid lattices

In the case of weakly modular matroid lattices we are able to give further necessary and sufficient conditions for $F_\aleph(L)$ to be a standard ideal.

The following theorem is a generalization of [6, Corollary 8] (cf. also [7, Corollary 5.2]) on arbitrary $F_\aleph(L)$.

Theorem 4.1. *Let L be a weakly modular matroid lattice of infinite length and let* $\aleph_0<\aleph\leqq r(L)$. *Then the following three conditions are equivalent:*

(i) $F_\aleph(L)$ *is a standard ideal of L;*

(ii) $F_\aleph(L)$ *is a p-ideal of L;*

(iii) $y<|z\Rightarrow y\in F_\aleph(L)$.

Proof. (i)⇒(ii)⇒(iii): these implications follow from Proposition 3.7.

(iii)⇒(i): by Theorem 3.6 it is sufficient to show that the implication

$$r[x, b\vee x] < \aleph \Rightarrow r[b\wedge x, x] < \aleph \tag{12}$$

is true. Let therefore

$$r[x, b\vee x] < \aleph. \tag{13}$$

We distinguish two cases: $b\wedge x>0$ and $b\wedge x=0$.

Let first $b\wedge x>0$. Then $[b\wedge x)$ is a modular matroid lattice and

$$b\wedge x,\ x,\ b,\ b\vee x\in[b\wedge x).$$

Since in a modular lattice transposed intervals are isomorphic, it follows that

$$r[b\wedge x,\ b]=r[x,\ b\vee x].$$

Because of (13) we have thus

$$r[b\wedge x,\ b]<\aleph.$$

This means that (12) is true in case $b\wedge x>0$.

Let now $b\wedge x=0$. Then $[0,\ b\vee x]$ is likewise a matroid lattice and by Proposition 3.1 there exists a maximal $m\in[0,\ b\vee x]$ for which

(14) $$x\leqq m<b\vee x$$

and

(15) $$b\wedge m=0$$

hold. By (14) it follows that there exists an atom $p\in L$ such that

(16) $$m<m\vee p\leqq b\vee x.$$

Then we obtain again by Proposition 3.1 that

(17) $$b\wedge(m\vee p)>0$$

and

(18) $$b\wedge(m\vee p)<\mid m.$$

From $x\leqq m\vee p\leqq b\wedge(m\vee p)\leqq b\vee x$ it follows by (13) that

(19) $$r[m\vee p,\ b\wedge(m\vee p)]<\aleph$$

holds. Moreover, the principal ideal $[b\wedge(m\vee p))$ is a modular lattice by (17) (for L is weakly modular) Thus we obtain

$$r[b\wedge(m\vee p),\ b]=r[m\vee p,\ b\vee(m\vee p)]$$

since transposed intervals of a modular lattice are isomorphie. By (19) we get from this that

(20) $$r[b\wedge(m\vee p),\ b]<\aleph$$

Furthermore we get from (18) by (iii) the relation

$$b\wedge(m\vee p)\in F_{\aleph}(L)$$

and thus

(21) $$r[0,\ b\wedge(m\vee p)]<\aleph.$$

(20) and (21) together yield[5])

$$r[b\wedge x,\ b]=r[0,\ b]\leqq r[0,\ b\wedge(m\vee p)]+r[b\wedge(m\vee p),\ b]<\aleph+\aleph=\aleph.$$

This proves (12) in case $b\wedge x=0$ and the proof is finished.

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[5]) It is not difficult to show the property of the rank used here.

Corollary 4.2. *Let L be a weakly modular matroid lattice of infinite length and let* $\aleph_0 \leqq \aleph \leqq r(L)$. *Then the following two conditions are equivalent:*

(i) *$F(L)$ is a standard ideal of L;*

(ii) *$F_\aleph(L)$ is a standard ideal of L for all* $(\aleph_0 \leqq \aleph \leqq r(L))$.

PROOF. (ii)$\Rightarrow$(i): if $F_\aleph(L)$ is a standard ideal for all $\aleph$ with $\aleph_0 \leqq \aleph \leqq r(L)$, then in particular, $F_{\aleph_0}(L) = F(L)$ is a standard ideal.

(i)$\Rightarrow$(ii): let now $F(L)$ be a standard ideal in L. Then by Proposition 4.1 the implication

$$y <| z \Rightarrow y \in F(L)$$

holds in L. Since $F(L) \subseteqq F_\aleph(L)$ for all $\aleph$, it follows that $y \in F_\aleph(L)$ for all $\aleph$ $(\aleph_0 \leqq \aleph \leqq r(L))$. Thus condition (iii) of Theorem 4.1 is satisfied. Condition (i) of Theorem 4.1 yields now that $F_\aleph(L)$ is a standard ideal for all $\aleph$ with $\aleph_0 \leqq \aleph \leqq r(L)$.

5. Non-modular affine matroid lattices

First we give some notions and definitions.

Axiom ("Euclid's weak parallel axiom", cf. [5, p. 78]). Let g be a line in a matroid lattice (that is, an element with $r(g)=2$). If p is an atom with $p \not\leqq g$, then there exists at most one line h with $g \parallel h$ and $p < h$.

Definition (cf. [5, Definition 18.3, p. 78]). Let L be a weakly modular matroid lattice with $r(L) \geqq 4$. If in L Euclid's weak parallel axiom holds, then L is called an *affine matroid lattice*.

Between non-modular affine matroid lattice and Wilcox lattices we have the following connection:

Theorem ([5, Corollary 19. 14, p. 90]). *If L is a non-modular affine matroid lattice, then L is a Wilcox lattice with imaginary unit.*[6])

Remark 5.1. If $L \equiv \Lambda - S$ is an AC-lattice, then Λ is also an AC-lattice ([5, Lemma 20.3, p. 91]). If L is a matroid lattice, then Λ is likewise a matroid lattice (for the atoms of L coincide with the atoms of Λ and the union of elements in L coincides with the union of these elements in Λ). It is therefore clear, what we mean by $F_\aleph(\Lambda)$. If L is a matroid lattice, then it follows by [5, Lemma 20.2 and Lemma 20.3, p. 91] that $a \in F_\aleph(L)$ holds if and only if $a \in F_\aleph(\Lambda)$. Moreover the rank $r(a)$ of a in L coincides with the rank of a in Λ.

For non-modular affine matroid lattices we generalize now [8, Theorem 4.3] on arbitrary $F_\aleph(L)$.

Theorem 5.2. *Let* $L \equiv \Lambda - S$ *be a non-modular affine matroid lattice of infinite length and let* $\aleph_0 \leqq \aleph \leqq r(L)$. *Then the following conditions are equivalent:*

(i) *$F_\aleph(L)$ is a standard ideal of L;*

(ii) *$F_\aleph(L)$ is a p-ideal of L;*

(iii) $y <| z \Rightarrow y \in F_\aleph(L)$;

(iv) $y \parallel z \Rightarrow y \in F_\aleph(L)$;

[6]) With the notations of § 2 we may therefore write $L \equiv \Lambda - S$.

(v) $S \subseteqq F_{\aleph}(\Lambda)$;
(vi) $i \in F_{\aleph}(\Lambda)$;
(vii) $M(L) \cap F_{\aleph}(L) \supset \{0\}$.[7])

PROOF. (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv): these implications follow from Proposition 3.7.

(iv)$\Rightarrow$(v): for the set S of the imaginary elements for L we have $S \subseteqq \Lambda - 0,1$. Hence if $u \in S$, then we have

$$u \neq 0, 1. \tag{22}$$

Choose an atom $p \in L$ for which

$$p \nleqq u. \tag{23}$$

Because of (22) this is always possible, since L is an AC-lattice. Consider now the element

$$a = p \vee u = p \sqcup u \tag{24}$$

(by [5, Theorem 3.11, p. 12] the union of two elements in L coincides with the union of these elements in Λ). By (22) and (23) it follows that $r(a) \geqq 2$ in L and in Λ (cf. Remark 5.1). The element $a \in L$ is by [5, Definition 21.1, p. 96] singular. We distinguish now the cases $a \neq 1$ and $a = 1$.
If $a \neq 1$, then there exists by [5, Lemma 21.7, p. 97] an element $b \in L$ such that $a \parallel b$. By (iv) it follows that $a \in F_{\aleph}(L)$ and from this $a \in F_{\aleph}(\Lambda)$ by Remark 5.1. By (24) we have $u \leqq a$ in Λ and therefore $u \in F_{\aleph}(\Lambda)$ holds.
If $a = 1$, then there exist by [5, Lemma 21.7, p. 97] two singular elements $a_1, a_2 \neq 1$ such that $a_1 \vee a_2 = 1$. Similarly as above, we obtain $a_1, a_2 \in F_{\aleph}(L)$ and thus $a_1 \vee a_2 = 1 \in F_{\aleph}(L)$. From this it follows by Remark 5.1 that $1 \in F_{\aleph}(\Lambda)$.

(v)$\Rightarrow$(iii): let $y <| z$. If y is an atom, then, of course, $y \in F_{\aleph}(L)$. If y is not an atom, then $y <| z$ implies by [5, Lemma 17.6, p. 72] the relation $(z, y)\overline{M}$. Moreover by [5, 3.11. 5, p. 12] $(z, y)\overline{M}$ in L holds if and only if $z \sqcap y \notin L$. Hence $z \sqcap y \in S$ and by condition (v), it follows that

$$z \sqcap y \in S \subseteqq F_{\aleph}(\Lambda). \tag{25}$$

By $z \prec z \vee y = z \sqcup y$ in Λ it follows that also $z \sqcap y \prec y$ because of the modularity of Λ. Thus we obtain by (25) that $y \in F_{\aleph}(\Lambda)$ and therefore $y \in F_{\aleph}(L)$ (cf. Remark (5.1).

(iii)$\Rightarrow$(i): this implication follows from Theorem 4.1.

(v)$\Rightarrow$(vi): let $S \Rightarrow F_{\aleph}(\Lambda)$. By [5, Corollary 19.14, p. 90] L has an imaginary unit i. Since $i \in S$, we get also $i \in F_{\aleph}(\Lambda)$.

(vi)$\Rightarrow$(v): this implication is true, since $u \leqq i$ for all $u \in S$.

(vi)$\Rightarrow$(viii): this implication follows from [5, Lemma 22.4, p. 104]. This proves the theorem.

[7]) $M(L)$ denotes the modular center of L (cf. [5, Definition 22.4, p. 104]).

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