On additive functions

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1. Let $f(n)$ and $g(n)$ denote additive arithmetical functions. Some years ago I proved that from

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqq x} |f(n+1)-f(n)| = 0 \tag{1.1}$$

it follows that $f(n)$ is a constant multiple of $\log n$ [1]. This was an old conjecture of P. ERDŐS [2]. Almost at the same time E. WIRSING [3] obtained a stronger result, namely that from

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{x\leqq n\leqq (1+\gamma)x} |f(n+1)-f(n)| = 0 \tag{1.2}$$

— γ being a positive constant — it follows that $f(n)=c\log n$.

The purpose of this paper is to generalize my previous result in two directions, which we state as Theorem 1 and 2.

Theorem 1. *Let $f(n)$ and $g(n)$ be additive functions and the relation*

$$\liminf_{x\to\infty} (\log x)^{-1} \sum_{n\leqq x} \frac{1}{n} |g(n+1)-f(n)| = 0 \tag{1.3}$$

hold. Then $f(n)=g(n)=c\log n$.

Let $\Delta^k f(n)$ denote the k'th *difference of $f(n)$, i.e. $\Delta^1 f(n)=f(n+1)-f(n)$, and generally*

$$\Delta^j f(n) = \Delta^{j-1} f(n+1) - \Delta^{j-1} f(n).$$

Theorem 2. *Let $f(n)$ be an additive function and for some positive integer k the relation*

$$\liminf_{x\to\infty} (\log x)^{-1} \sum_{n\leqq x} \frac{|\Delta^k f(n)|}{n} = 0 \tag{1.4}$$

hold. Then $f(n)$ is a constant multiple of $\log n$.

Namely for $k=1$ Theorem 2 gives that $f(n)$ is a constant multiple of $\log n$, if

$$\liminf_{x\to\infty} (\log x)^{-1} \sum_{n\leqq x} \frac{|f(n+1)-f(n)|}{n} = 0. \tag{1.5}$$

This a little stronger assertion then that was published in [1] but weaker than the cited result due to Wirsing.

First we prove Theorem 2 in the case $k=1$. After then we prove that from the condition (1.4) it follows that it holds for $k-1$ instead of k. Finally we prove Theorem 1 by showing that from (1.3) $f(n)=g(n)$ follows.

2. Proof of Theorem 2 for $k=1$

We need a lemma.

Lemma 1. *If* (1.5) *holds then* $f(n)$ *is completely additive.*

PROOF. Let

$$g_a(n) = \max_{-a\leqq j\leqq a} |f(n+j)-f(n)|. \tag{2.1}$$

Then from (1.5) we get easily that

$$\liminf_{x\to\infty} \frac{1}{x} \sum_{n\leqq x} g_a(n) = 0, \tag{2.2}$$

for every fixed a.

We need to prove that $f(p^\nu)=\nu f(p)$ for all prime-power p^ν. For an arbitrary N coprime to p we have

$$f(p^\nu)+f(N) = f(p^\nu N) = \{f(p^\nu N)-f(p^\nu N+p)\}+f(p)+f(p^{\nu-1}N+1) = \dots$$

$$\dots = \{f(p^\nu N)-f(p^\nu N+p)\}+\sum_{j=1}^{\nu-2}\{f(p^{\nu-j}N+1)-f(p^{\nu-j}j+p)\}+$$

$$+\{f(pN+1)-f(pN)\}+f(N)+\nu f(p).$$

Hence

$$|f(p^\nu)-\nu f(p)| \leqq g_p(p^\nu N)+g_p(p^{\nu-1}N)+\dots+g_p(pN),$$

and so

$$|fp^\nu-\nu f(p)|\cdot \sum_{\substack{N\leqq x\\(N,\,p)=1}} 1 \leqq \sum_{m\leqq xp} g_p(m).$$

Since

$$\sum_{\substack{N\leqq x\\(N,\,p)=1}} 1 = x(1-1/p)+O(1),$$

by (2.2) we have $f(p^\nu)=\nu f(p)$. This completes the proof of our lemma.

Let $x_\nu=p^\nu$ $(\nu=0, 1, 2, \dots)$ where p is an arbitrary prime. Let for $x_\nu\leqq N<x_{\nu+1}$

$$N = \sum_{j=0}^{\nu} a_j(N)p^{\nu-j} \tag{2.3}$$

be the p-adical representation of N. Then

$$0 \leqq a_j(N) \leqq p-1 \quad (0\leqq j\leqq \nu), \quad a_0(N)\geqq 1.$$

For an arbitrary N let the sequence $(N=)N_0, N_1, \ldots, N_\nu$ be defined as follows:

$$N_k = \sum_{j=0}^{\nu-k} a_j(N)\, p^{\nu-j-k} \quad (k = 0, \ldots, \nu). \tag{2.4}$$

So we have

$$N_{k-1} = pN_k + a_{\nu-k+1}(N) \quad (k = 1, \ldots, \nu). \tag{2.5}$$

Since

$$f(N_0) = \{f(N_0) - f(pN_1)\} + \{f(N_1) - f(pN_2)\} + \ldots$$
$$+ \{f(N_{\nu-1}) - f(pN_\nu)\} + f(N_\nu) + \nu f(p),$$

we have

$$|f(N_0) - \nu f(p)| \leqq g_p(N_0) + g_p(N_1) + \ldots + g_p(N_{\nu-1}) + A_p,$$

where

$$A_p = \max_{j=1,\ldots,p-1} |f(j)|.$$

For $N \in (x_\nu, x_{\nu+1})$ we get $N_k \in [x_{\nu-k}, x_{\nu-k+1})$. Furthermore, any fixed m in $[x_{\nu-k}, x_{\nu+1-k})$ occurs exactly p^k times as N_k. Therefore

$$\frac{1}{x_{\nu+1}} \sum_{x_\nu \leqq N < x_{\nu+1}} |f(N) - \nu f(p)| \leqq \frac{1}{x_{\nu+1}} \sum_{x_\nu \leqq N < x_{\nu+1}} g_p(N) +$$
$$+ \frac{1}{x_\nu} \sum_{x_{\nu-1} \leqq N < x_\nu} g_p(N) + \ldots + A_p \leqq \sum_{N \leqq x_{\nu+1}} \frac{g_p(N)}{N} + A_p.$$

Since $\nu = \left[\frac{\log N}{\log p}\right]$ for $x_\nu \leqq N < x_{\nu+1}$, therefore we have

$$\frac{1}{x_{\nu+1}} \sum_{x_\nu \leqq N < x_{\nu+1}} \left| \frac{f(N)}{\log N} - \frac{f(p)}{\log p} \right| \leqq \frac{c_p}{\log x_{\nu+1}} \sum_{N \leqq x_{\nu+1}} \frac{g_p(N)}{N}. \tag{2.6}$$

c_p and later c_q are suitable constants.

Similarly, when q is another prime, $y_\mu = q^\mu$, then

$$\frac{1}{y_{\mu+1}} \sum_{y_\mu \leqq N < y_{\mu+1}} \left| \frac{f(N)}{\log N} - \frac{f(q)}{\log q} \right| \leqq \frac{c_q}{\log y_{\mu+1}} \sum_{N \leqq y_{\mu+1}} \frac{g_q(N)}{N}.$$

Let $q < p$. Then the interval $(x_\nu, x_{\nu+1})$ contains an integer-power of q, say y_μ. Then there is at least one subinterval I_ν in $[x_\nu, x_{\nu+1})$ with length greater than $cx_{\nu+1}$ (c being a positive constant) the endpoints of which are powers of p and q. Then by (2.5) and (2.6)

$$\frac{1}{x_{\nu+1}} \sum_{N \in I_\nu} \left| \frac{f(p)}{\log p} - \frac{f(q)}{\log q} \right| \leqq \frac{c_1}{\log x_{\nu+1}} \sum_{N \leqq x_{\nu+1}} \frac{g_p(N)}{N}.$$

Since

$$\liminf_\nu (\log x_{\nu+1})^{-1} \sum_{N \leqq x_{\nu+1}} N^{-1} g_p(N) = 0 \quad \text{and} \quad \sum_{N \in I_\nu} 1 \geqq cx_{\nu+1},$$

we have

$$\frac{f(p)}{\log p} = \frac{f(q)}{\log q}.$$

This completes the proof of our assertion.

3. Reduction step

Let H_k denote the assertion stated in (1.4). We prove that H_k implies H_{k-1} for $k \geqq 2$.

Let $\Delta_2 f(n) = f(n+2) - f(n)$, and

$$\Delta_2^j f(n) = \Delta_2^{j-1} f(n+2) - \Delta_2^{j-1} f(n). \tag{3.1}$$

Lemma 2. *For every n*

$$\Delta_2^k f(n) = \sum_{j=0}^{k} \binom{k}{j} \Delta^k f(n+j), \tag{3.2}$$

and for every odd n

$$\Delta_2^k f(n) = \sum_{j=0}^{k} \binom{k}{j} \Delta_2^k f(2n+2j) \tag{3.3}$$

hold.

The proof of these relations is straightforward so we omit it.

From (3.2) we get — using it by $k-1$ instead of k —

$$\Delta_2^{k-1} f(n+1) - \Delta_2^{k-1} f(n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \Delta^k f(n+j). \tag{3.4}$$

Furthermore from (3.2) we get the relation

$$|\Delta_2^k f(n)| \leqq \sum_{j=0}^{k} \binom{k}{j} |\Delta^k f(n+j)|. \tag{3.5}$$

Hence

$$|\Delta_2^{k-1} f(n+2) - \Delta_2^{k-1} f(n)| \leqq 2^{k+1} \sum_{j=0}^{k} |\Delta^k f(n+j)|. \tag{3.6}$$

From (3.3) we get for odd n — using it by $k-1$ instead of k —

$$\Delta_2^{k-1} f(n) - 2^{k-1} \Delta_2^{k-1} f(2n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \{\Delta_2^{k-1} f(2n+2j) - \Delta_2^{k-1} f(2n)\},$$

and hence

$$|\Delta_2^{k-1} f(n) - 2^{k-1} \Delta_2^{k-1} f(2n)| \leqq 2^{k-1} \sum_{h=0}^{k-1} |\Delta_2^k f(2n+2h)|. \tag{3.8}$$

Let

$$x_\nu = 2^\nu \quad (\nu = 1, 2, \ldots), \quad I_\nu = [x_\nu, x_{\nu+1})$$

and

$$\alpha(\nu) = \sum_{n \in I_\nu} |\Delta^{k-1} f(n+1) - \Delta^{k-1} f(n)| = \sum_{n \in I_\nu} |\Delta^k f(n)|. \tag{3.9}$$

Let B, C, E denote the set of even, odd, two-times-odd integers, respectively. Correspondingly let $\beta(\nu)$, $\gamma(\nu)$, $\varepsilon(\nu)$ the sum

$$\sum_n |\Delta_2^{k-1} f(n)|$$

where we take the summation over the sets $B \cap I_\nu$, $C \cap I_\nu$, $E \cap I_\nu$, respectively.

From (3.6) and (3.2) we get

$$\gamma(\nu+1) = \sum_{2m \in I_{\nu+1}} |\Delta_2^{k-1} f(2m)| \leqq 2 \sum_{\substack{m \in I_\nu \\ m \in B}} |\Delta_2^{k-1} f(2m)| + 2 \sum_{m \in I_\nu} |\Delta_2^k f(2m)| \leqq$$

$$\leqq 2\varepsilon(\nu+1) + 2^{2k+1} \sum_{x_{\nu+1} \leqq m < x_{\nu+2}+k} |\Delta^k f(m)|.$$

Assume now that $k < x_{\nu+1}$. Then

$$\gamma(\nu+1) \leqq 2\varepsilon(\nu+1) + 2^{2k+1}\big(\alpha(\nu+1) + \alpha(\nu+2)\big). \tag{3.10}$$

From the inequality (3.8) we get

$$\varepsilon(\nu+1) = \sum_{\substack{m \in I_\nu \\ m \in B}} |\Delta_2^{k-1} f(2m)| \leqq \frac{1}{2^{k-1}} \sum_{\substack{m \in I_\nu \\ m \in B}} |\Delta_2^{k-1} f(m)| +$$

(3.11)

$$+ k \sum_{x_\nu \leqq m \leqq x_{\nu+2}-2k} |\Delta_2^k f(2m)| \leqq \frac{1}{2^{k-1}} \beta(\nu) + 2^k k\big(\alpha(\nu+1) + \alpha(\nu+2)\big).$$

So by (3.10) we get

$$\gamma(\nu+1) \leqq \frac{1}{2^{k-2}} \beta(\nu) + 2^{2k+2}\big(\alpha(\nu+1) + \alpha(\nu+2)\big). \tag{3.12}$$

Similarly from (3.4) we get easily that

$$|\beta(\nu+1) - \gamma(\nu+1)| \leqq 2^{k+1}\big(\alpha(\nu+1) + \alpha(\nu+2)\big). \tag{3.13}$$

So we have

$$\beta(\nu+1) \leqq \frac{1}{2^{k-2}} \beta(\nu) + c_1(k)\big(\alpha(\nu+1) + \alpha(\nu+2)\big) \tag{3.14}$$

if $k < x_{\nu+1}$, where $c_1(k)$ is a constant that depends only on k.
From the condition (1.4) it follows that

$$\liminf_{\mu} \frac{1}{\mu} \sum_{\nu \leqq \mu} \frac{\alpha(\nu)}{x_\nu} = 0. \tag{3.15}$$

Let $\varrho_\nu = \dfrac{\beta(\nu)}{2^\nu}$. From (3.15) we have

$$\varrho_{\nu+1} \leqq \frac{1}{2^{k-1}} \varrho_\nu + \tau_\nu, \quad (x_\nu \geqq k) \tag{3.16}$$

where

$$\tau_\nu = c_2(k) \left(\frac{\alpha(\nu+1)}{2^{\nu+1}} + \frac{\alpha(\nu+2)}{2^{\nu+2}} \right).$$

Let $\mu_1 < \mu_2 < \dots$ be a sequence of integers so that

$$\frac{1}{\mu_t} \sum_{\nu \leqq \mu_t} \tau_\nu \to 0 \quad (t \to \infty). \tag{3.17}$$

From (3.15) we can deduce easily that

$$\lim_{t\to\infty} \frac{1}{\mu_t} \sum_{\nu \leqq \mu_t} \varrho_\nu = 0.$$

Observing (3.13), we have

$$\lim_{x_{\mu_t}\to\infty} (\log x_{\mu_t})^{-1} \sum_{n \leqq x_{\mu_t}} \frac{|\Delta_2^{k-1} f(n)|}{n} = 0. \tag{3.18}$$

From (3.2) we get

$$\Delta_2^{k-1} f(n) - 2^{k-1}\Delta^{k-1} f(n) = \sum_{j=0}^{k-1} \binom{k-1}{j} \{\Delta^{k-1} f(n+j) - \Delta^{k-1} f(n)\} =$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j} \left\{ \sum_{h=0}^{j-1} \Delta^k f(n+h) \right\},$$

and hence

$$|\Delta^{k-1} f(n)| \leqq \frac{1}{2^{k-1}} |\Delta_2^{k-1} f(n)| + k \cdot \sum_{j=0}^{2k-3} |\Delta^k f(n+h)|,$$

and so

$$\sum_{n\leqq x} \frac{|\Delta^{k-1} f(n)|}{n} \leqq \frac{1}{2^{k-1}} \sum_{n\leqq x} \frac{|\Delta_2^{k-1} f(n)|}{n} + 2k^2 \sum_{n \leqq x+2k} \frac{|\Delta^k f(n)|}{n}.$$

Let now x be choosen as $x = x_{\mu_t} - 2k$. Then from (3.17) and (3.18) we get

$$\lim_{t\to\infty} (\log x_{\mu_t})^{-1} \sum_{n \leqq x_{\mu_t} - 2k} \frac{|\Delta^{k-1} f(n)|}{n} = 0. \tag{3.19}$$

So we proved that the condition H_{k-1} holds.
By this, Theorem 2 has been proved.

4. Proof of Theorem 1

We need to prove only that from (1.3) the relation $f(n) = g(n)$ follows. Let $\varrho(n) = g(n+1) - f(n)$, and $H(n) = g(n) - f(n)$. We observe that

$$g(16k+12) - f(16k+10) = [g(4) - g(2) - f(2)] + \varrho(8k+5) \tag{4.1}$$

and that

$$g(16k+12) - f(16k+10) = \varrho(16k+11) - H(16k+11) + \varrho(16k+10). \tag{4.2}$$

Let $C = g(4) - g(2) - f(2)$.
From (4.1) and (4.2) we have

$$C + H(16k+11) = \varrho(16k+11) + \varrho(16k+10) - \varrho(8k+5). \tag{4.3}$$

Let $m \equiv 1 \bmod 16$ be an arbitrary but fixed integer, and $16k+11$ coprime to m. Then from (4.3)

$$C+H(m(16k+11)) = \varrho(m(16k+11))+\varrho(m(16k+11)-1)-\varrho\left(\frac{m(16k+11)-1}{2}\right),$$

and so

$$H(m) = \varrho(m(16k+11))+\varrho(m(16k+11)-1)-\varrho\left(\frac{m(16k+11)-1}{2}\right)-$$

$$-\varrho(16k+11)-\varrho(16k+10)+\varrho(8k+5).$$

Hence we have

$$|H(m)| \sum_{\substack{k \leqq x \\ (k,m)=1}} \frac{1}{k} \leqq 6 \sum_{n \leqq 18x} \frac{\varrho(n)}{n}.$$

Observing that

$$\lim_{x\to\infty} (\log x)^{-1} \sum_{\substack{k \leqq x \\ (k,m)=1}} 1/k > 0,$$

and (1.3), we obtain that $H(m)=0$.

Let $m_1 \equiv m_2 \pmod{16}$ be odd integers. We can choose a ν so that $(m_1 m_2, \nu)=1$ and $m_i \nu \equiv 1 \pmod{16}$. Then $H(m_1)+H(\nu)=H(m_2)+H(\nu)=0$, whence $H(m_1)= =H(m_2)$. We have that the value of $H(m)$ depends only on the residue class $m \pmod{16}$. But this is possible only if $H(m)=0$ for every odd n.

Let $n \equiv 1 \bmod 3$. Then

$$\varrho(n) = \varrho(3n+2)+\varrho(3n+1)+\varrho(3n)-H(3n+2)-H(3n+1). \tag{4.4}$$

Let B_α denote the set of those integers n, for which

$$n \equiv 1 \pmod 3, \quad 2^\alpha \| 3n+1$$

hold. For $n \in B_\alpha$ $(\alpha \geqq 1)$ $H(3n+2)=0$. From (4.4) we get

$$|H(2^\alpha)| \sum_{\substack{n \leqq x \\ n \in B_\alpha}} \frac{1}{n} \leqq 3 \sum_{m \leqq 4x} \frac{|\varrho(m)|}{m}.$$

Observing that

$$\lim_{x\to\infty} (\log x)^{-1} \sum_{\substack{n \leqq x \\ n \in B_\alpha}} \frac{1}{n} > 0,$$

from the relation (1.3) $H(2^\alpha)=0$ follows. Consequently $H(n)=0$ identically, and so Theorem 1 is a consequence of Theorem 2.

References

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