On (m, n)-ideals in associative rings

By S. LAJOS and F. SZÁSZ

1. Introduction

We introduce in this paper the following generalizations of the concepts of ideals in rings: the notion of (m, n)-ideal, which is a generalization of one-sided (left or right) ideals and it contains, as a particular case, the notion of bi-ideal due to R. A. GOOD and D. R. HUGHES [3] for semigroups, and due to the authors [17] for rings; the concept of (m, n)-quasi-ideal which is a generalization of the concept of quasi-ideal due to O. STEINFIELD; the concept of i^k-ideal and k-ideal, the latter is a generalization of the concept of two-sided ideal.

Let us point out that (m, n)-ideals, (m, n)-quasi-ideals, i^k-ideals, and k-ideals were before introduced and investigated by the first author (see [6], [8], [12], [15]). For i^k-ideals, see also R. BAER [2].

For the necessary background notions used throughout this paper, we refer to N. JACOBSON [4], S. LAJOS [6], N. H. MCCOY [19], and F. SZÁSZ [23].

By a ring we mean an associative ring. Throughout this paper A always denotes an associative ring.

For the additive subgroups B and C of an associative ring A let the product BC denote the additive subgroup of the additive group A^+ of A, generated by all the product bc with $b \in B$ and $c \in C$. Let us remark that this complex multiplication of the additive subgroups is again associative, i.e. the relation

$$B(CD) = (BC)D$$

holds for arbitrary additive subgroups B, C, D of A^+. Let furthermore $S^1 = S$ and $S^n = S^{n-1}S$ for an arbitrary subring S of A (S^0 is defined as an operator element such that $S^0A = AS^0 = A$). I always denotes the ring of rational integers and thus IX, for an arbitrary non-empty subset X of A, denotes the additive subgroup of A^+ generated by X.

In Section 2 we develop the fundamentals of the theory of (m, n)-ideals of associative rings. First we assert some important properties of (m, n)-ideals in rings. Furthermore, Theorem 2.4 gives a sufficient condition for a product BC, where B is an (m, n)-ideal and C is an additive subgroup of A^+, to be again an (m, n)-ideal of A. In the same section we introduce the π-ideals of a ring, and we show in Theorem 2.7 that every lr-ideal is an $(1, 1)$-ideal and conversely. More generally, our Theorem 2.9 asserts that an additive subgroup of A^+ is a π-ideal of A if and only if it is an (m, n)-ideal of A for certain non-negative integers m and n. Theorem 2.10 gives another sufficient condition for a product BC to be an (m, n)-ideal, provided that B is an $(m, 0)$-ideal and C is a $(0, n)$-ideal of the ring A. Corollary 2.16 states that i^k-ideals and k-ideals coincide if the ring A is commutative. Finally, in Proposition 2.17 we point out that a ring A contains no proper (m, n)-ideals if and only if it is a divison ring.

In Section 3 we discuss (m, n)-ideals of rings with minimum condition on subrings. Thus our Theorem 3.1 states that in the case $m, n \geqq 2$ a ring A with minimum condition on subrings contains either a proper $(1, k)$-ideal or else a proper $(k, 1)$-ideal. Corollary 3.2 is the application of this result to finite rings.

In Section 4 we define the (m, n)-quasi-ideals of rings and investigate them. First we assert several fundamental properties. Then, in Theorem 4.2, we show that every (m, n)-quasi-ideal is an (m, n)-ideal of the ring. Theorem 4.4 asserts that in any ring with a distributive lattice of subrings, a subring is an (m, n)-quasi-ideal if and only if it is the intersection of an $(m, 0)$-ideal and a $(0, n)$-ideal of the ring.

In Section 5 we prove that in a von Neumann regular ring A every (m, n)-ideal is an (m, n)-quasi-ideal and conversely. Corollary 5.6 states that every i^k-ideal of a von Neumann regular ring A is a two-sided ideal of A.

2. (m, n)-ideals in associative rings

Definition 2.1. Let A be an associative ring, S a subring of A and m, n non-negative integers. We say that S is an *(m, n)-ideal* of A *if* $S^m A S^n \subseteqq S$.

The present authors [17] defined the notion of $(1, 1)$-ideal of rings under the name "bi-ideal" (cf. also for semigroups R. A. Good and D. R. Hughes [3]).

It is easy to prove the following statements concerning (m, n)-ideals of rings.

a) The intersection of any two (m, n)-ideals of a ring A is again an (m, n)-ideal of A.

b) A divison ring has no proper (m, n)-ideals.

c) The power S^k of an (m, n)-ideal S is again an (m, n)-ideal, where k is an arbitrary positive integer.

Let B be a non-empty subset of the ring A. Then the smallest (m, n)-ideal of A containing B is called *the (m, n)-ideal of A generated by B* and denoted by $\{B\}_{(m,n)}$.

d) It is clear that

$$\{B\}_{(m,n)} = IB + IB^2 + \ldots + IB^{m+n} + IB^m AB^n.$$

e) If B is a subring of a ring A then

$$\{B\}_{(m,n)} = B + B^m AB^n.$$

f) The principal (m, n)-ideal of a ring A generated by the element a of A is the following:

$$\{a\}_{(m,n)} = Ia + Ia^2 + \ldots + Ia^{m+n} + Ia^m Aa^n.$$

g) The principal (m, n)-ideal of A generated by an idempotent element e is

$$\{e\}_{(m,n)} = \begin{cases} eA, & \text{if } m \neq 0, \quad n = 0; \\ eAe, & \text{if } m \neq 0, \quad n \neq 0; \\ Ae, & \text{if } m = 0, \quad n \neq 0. \end{cases}$$

Theorem 2.2. *Let A be a ring, S a subring of A and T an (m, n)-ideal of A. Then the intersection $S \cap T$ is an (m, n)-ideal of S.*

PROOF. The intersection $S \cap T$ obviously is a subring of S. We show that

$$(S \cap T)^m S(S \cap T)^n \subseteqq S \cap T.$$

Since T is an (m, n)-ideal of A, we have

$$(S \cap T)^m S(S \cap T)^n \subseteqq T.$$

On the other hand, S is a subring of A. Hence

$$(S \cap T)^m S(S \cap T)^n \subseteqq S.$$

Therefore $S \cap T$ is an (m, n)-ideal of S.

Corollary 2.3. *Let B be an (m, n)-ideal, C a (p, q)-ideal of a ring A. Then the intersection $B \cap C$ is a (p, q)-ideal of the ring B and an (m, n)-ideal of the ring C.*

Theorem 2.4. *Let A be a ring, B an (m, n)-ideal of A and C an additive subgroup of A^+ satisfying either $BC \subseteqq B$ or $CB \subseteqq B$. Then both the products BC and CB are (m, n)-ideals of the ring A.*

PROOF. Assume, for instance, $BC \subseteqq B$. Then the product BC is an additive subgroup of A^+ by the definition of BC. Furthermore we have

$$(BC)(BC) \subseteqq B \cdot BC \subseteqq BC,$$

thus the product BC is a subring of A. On the other hand,

$$(BC)^m A(BC)^n \subseteqq B^m AB^{n-1}(BC) \subseteqq BC,$$

whence BC is an (m, n)-ideal of the ring A.

It can similarly be proved that the product CB is also an (m, n)-ideal of the ring A.

Corollary 2.5. *Let m, n be positive integers, A a ring, S an (m, n)-ideal of A and $s \in S$. Then the products sS and Ss are (m, n)-ideals of A.*

Definition 2.6. A subring S_n of a ring A is said to be *attainable* if there exists a sequence $S_1, S_2, \ldots, S_{n-1}$ of subrings of A such that

$$S_n \subseteqq S_{n-1} \subseteqq \ldots \subseteqq S_1 \subseteqq S_0 = S$$

holds, where S_i is a one-sided (left or right) ideal of S_{i-1} $(i=1, 2, \ldots, n)$.

To every chain of such subrings of A we can associate a product π of the letters l and r whose i-th factor is l or r according to the fact that S_i is contained in S_{i-1} as a left or right ideal, respectively $(i=1, 2, \ldots, n)$. If S_i is a two-sided ideal of S_{i-1}, then any of l and r can be chosen. Furthermore, a subring S of A is called a *π-ideal*, if it is attainable by a subring chain to which the product π is associated. In the product π let m and n be the numbers of the factors r and l, respectively.

We are going to prove the following result.

Theorem 2.7. *For an additive subgroup S of a ring A the following statements are pairwise equivalent:*

(i) *S is an lr-ideal of A.*

(ii) *S is an rl-ideal of A.*

(iii) *S is an $(1, 1)$-ideal of A.*

PROOF. Suppose that S is an lr-ideal of a ring A. Then there exists a subring L of A such that $S \subseteqq L \subseteqq A$, $SL \subseteqq S$, and $AL \subseteqq L$. Hence

$$SAS \subseteqq SAL \subseteqq SL \subseteqq S,$$

that is, S is an $(1, 1)$-ideal of A, indeed.

Conversely, let S be an $(1, 1)$-ideal of the ring A, that is,

$$SAS \subseteqq S.$$

Then we have

$$S(S+AS) = S^2+SAS \subseteqq S+S = S.$$

It follows that S is a right ideal of the left ideal $S+AS$ of A. Consequently, S is an lr-ideal of A.

The proof of the left-right dual statement is analogous.

Corollary 2.8. *An additive subgroup S of a ring A is a π-ideal of A if and only if S is an $r^m l^n$-ideal of A.*

This follows at once from the equivalence of the conditions (i), (ii) in Theorem 2.7.

Now we are ready to prove the following result.

Theorem 2.9. *An additive subgroup S of a ring A is a π-ideal of A if and only f S is an (m, n)-ideal of A.*

PROOF. By the preceding Corollary 2.8 it suffices to verify our statement for $r^m l^n$-ideals instead of π-ideals. Let S be an $r^m l^n$-ideal of a ring A. Then S is an attainable subring of A, that is, there exist subrings $L_1, L_2, \ldots, L_{n-1}$ and $R_1, R_2, \ldots, R_m$ of A such that the following relations hold:

$$S = L_n \subseteqq L_{n-1} \subseteqq \ldots \subseteqq L_1 \subseteqq R_m \subseteqq \ldots \subseteqq R_1 \subseteqq R_0 = A,$$

$$R_i R_{i-1} \subseteqq R_i, \quad R_m L_1 \subseteqq L_1, \quad L_{j-1} L_j \subseteqq L_j$$

$$(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n).$$

This yields at once

$$S^m A S^n = L_n^m A L_n^n \subseteqq L_n^{m-1}(R_1 A) L_n^n \subseteqq L_n^{m-1} R_1 L_n^n \subseteqq L_n^{m-2}(R_2 R_1) L_n^n \subseteqq L_n^{m-2} R_2 L_n^n \subseteqq$$

$$\subseteqq \ldots \subseteqq (L_n R_{m-1}) L_n^n \subseteqq (R_m R_{m-1}) L_n^n \subseteqq R_m L_n^n \subseteqq$$

$$\subseteqq (R_m L_1) L_n^{n-1} \subseteqq L_1(L_2 L_n^{n-2}) \subseteqq \ldots \subseteqq L_n = S,$$

therefore S is an (m, n)-ideal of the ring A, indeed.

Conversely, let us suppose that S is an (m, n)-ideal of a ring A. Then, by the property *e)*, the (p, q)-ideal of A generated by S is $S+S^p A S^q$. Now it is easy to see, that $S_{(m,k)}$ is a left ideal of $S_{(m,k-1)}$ $(k=1, \ldots, n)$, and $S_{(i,n)}$ is a right ideal of

$S_{(i-1,n)}$ $(i=1, \dots, m)$. Hence the subrings $L_n=S$, $L_{n-1}=S_{(m,n-1)}, \dots, L_1=S_{(m,1)}$, $R_m=S_{(m,0)}$, $R_{m-1}=S_{(m-1,0)}, \dots, R_1=S_{(1,0)}$ satisfy the above conditions. Thus S is an $r^m l^n$-ideal of the ring A, which completes the proof of Theorem 2.9.

Theorem 2.10. *Let m, n be positive integers, B an $(m, 0)$-ideal and C a $(0, n)$-ideal of a ring satisfying the condition $BC=CB$. Then the product BC is an (m, n)-ideal of A.*

PROOF. Obviously, the condition $BC=CB$ yields

$$(BC)^2 = B^2C^2 \subseteqq BC,$$

thus the product BC is a subring of A. On the other hand, we have

$$(BC)^m A (BC)^n = B^m (C^m A B^n) C^n \subseteqq (B^m A) C^n \subseteqq BC,$$

consequently BC is an (m, n)-ideal of A, indeed.

Remark 2.11. In the particular case $m=n=1$ the assumption $BC=CB$ is superfluous. Namely, if L is a left ideal and R is a right ideal of a ring A, then the products LR and RL are $(1, 1)$-ideals of A. (Moreover, LR is a two-sided ideal of A.)

Definition 2.12. A two-sided ideal of a two-sided ideal of a ring A will be called an i^2-ideal of A. By an *i^k-ideal* we mean a two-sided ideal of an arbitrary i^{k-1}-ideal of A, where k is a positive integer $(k \geqq 2)$.

Remark 2.13. The i^k-ideals for $k=2, 3, 4, \dots$ are called also accessible subrings. Furthermore, R. BAER [2] calls these subrings "metaideals of finite index".

Definition 2.14. The subring S of a ring A is called a *k-ideal* of A, if S is an (m, n)-ideal of A for every m, n such that $m+n=k$.

Remark 2.15. It is clear that a subring S of a commutative ring A is a k-ideal of A if and only if the condition

$$S^k A \subseteqq S$$

holds. We observe that the concept of the k-ideal is a generalization of the concept of two-sided ideal.

Corollary 2.16. *A subring S of a commutative ring A is an i^k-ideal of A if and only if it is a k-ideal of A.*

PROOF. This follows at once from our Theorem 2.9.

Proposition 2.17. *A ring A contains no proper (m, n)-ideal if and only if it is a division ring.*

PROOF. The sufficiency is obvious. The necessity proof is the following. For every element $a \in A$ the subrings aA and Aa are (m, n)-ideals of A. Therefore

$$aA = Aa = A,$$

which implies that A is a division ring.

3. (m, n)-ideals in rings with minimum cordition on subrings

As it is well known, the rings with minimum condition on subrings have been studied by V. I. Šnejdmüller [21].

Next we prove the following result.

Theorem 3.1. *Let A be a ring with minimum condition on subrings. If A contains a proper (m, n)-ideal with $m \geqq 2$ and $n \geqq 2$, then A contains also a proper $(1, k)$-ideal or a proper $(k, 1)$-ideal for some $k \geqq 2$.*

Proof. Let m_1 denote the smallest positive integer for which A contains a proper (m_1, n)-ideal, and n_1 the smallest positive integer for which A contains a proper (m, n_1)-ideal. Then, by our assumptions, $m, n, m_1, n_1 \geqq 2$. We show that either $m_1 \leqq n$ or $n_1 \leqq m$ holds. Suppose that, in contrary, we have both of $m_1 > n$ and $n_1 > m$. Then, by $m_1 \leqq m$, we obtain $n_1 > m \geqq m_1 > n$, which is a contradiction with respect to the definition of n_1.

Thus we may assume, for instance, that $2 \leqq m_1 \leqq n$. Furthermore, let S be a proper (m_1, n)-ideal of the ring A. Now we define the following sequence of subrings of A:

$$B_1 = S^m A S^n, \quad B_{i+1} = B_i^{m_1} A B_i^n \quad (i = 1, 2, \ldots).$$

We can verify that $B_i^2 \subseteq B_i$. It is easy to see that

$$B_1 \supseteqq B_2 \supseteqq B_3 \supseteqq \ldots \quad \text{and} \quad B_i^{m_1} A B_i^n \subseteqq S.$$

Now the minimum condition on subrings of A implies

$$B_j = B_{j+1} = B_{j+2} = \ldots = B$$

for a fixed positive integer j. Then we have

$$B = B^{m_1} A B^n,$$

whence it follows at once

$$B^{m_1} A B^{n-m_1} B^{m_1} A B^n = BAB^n.$$

Therefore we obtain

$$B^{m_1} A B^{n-m_1+1} = BAB^n$$

as well. Postmultiplying this equality by B^{m_1-1}, we arrive

$$B^{m_1} A B^{n-m_1+1} B^{m_1-1} = BAB^n \cdot B^{m_1-1}.$$

Finally we obtain

$$B = BAB^{n+m_1-1}.$$

In other words, the subring B of A is an $(1, k)$-ideal of A with $k = n + m_1 - 1$.

One can prove in a similar way, that in the case of $n_1 \leqq m$ there exists a proper $(k, 1)$-ideal of the ring A.

Corollary 3.2. *If a finite ring A contains a proper (m, n)-ideal with $m \geqq 2$ an $n > 2$, then A contains also a proper $(1, k)$-ideal, or else a proper $(k, 1)$-ideal of A.*

4. (m, n)- quasi-ideals of rings

Definition 4.1. A subring S of a ring A will be called an *(m, n)-quasi-ideal* of A if the inclusion

$$S^m A \cap AS^n \subseteq S$$

holds, where m, n are non-negative integers (S^0 is an operator element not contained in A and such that $S^0 A = AS^0 = A$).

It is easy to prove the following properties of (m, n)-quasi-ideals:

a) The intersection of a set of (m, n)-quasi-ideals of a ring A is again an (m, n)-quasi-ideal of A.

b) A division ring has no proper (m, n)-quasi-ideal.

c) Let B be a subset of a ring A. Then the (m, n)-quasi-ideal of A generated by B, i.e., the smallest (m, n)-quasi-ideal of A containing B is

$$IB + IB^2 + \ldots + IB^k + (IB^m A \cap IAB^n),$$

where $k = \max(m, n)$.

d) If S is a subring of a ring A, then the (m, n)-quasi-ideal of A generated by S is

$$S + (S^m A \cap AS^n).$$

e) The principal (m, n)-quasi-ideal generated by the element a of a ring A is

$$Ia + Ia^2 + \ldots + Ia^k + (a^m A \cap Aa^n),$$

where $k = \max(m, n)$.

f) The principal (m, n)-quasi-ideal generated by an idempotent element e of A is

$$e^m A \cap Ae^n = \begin{cases} Ae & \text{if} \quad m = 0, \quad n \neq 0; \\ eA \cap Ae & \text{if} \quad m \neq 0, \quad n \neq 0; \\ eA & \text{if} \quad m \neq 0, \quad n = 0. \end{cases}$$

Theorem 4.2. *Every (m, n)-quasi-ideal of a ring A is an (m, n)-ideal of A.*

PROOF. Let A be a ring and S an (m, n)-quasi-ideal of A. Since we have $S^m AS^n \subseteq S^m A$ and $S^m AS^n \subseteq AS^n$, we obtain

$$S^m AS^n \subseteq S^m A \cap AS^n \subseteq S,$$

that is, S is an (m, n)-ideal of the ring A.

Remark 4.3. The concept of (1, 1)-quasi-ideal was introduced by O. STEINFELD [22] under the name "quasi-ideal". He showed that a subset of a semigroup is a quasi-ideal if and only if it is the intersection of a left ideal and a right ideal. The corresponding problem for associative rings is yet open. It has been answered for non-associative rings in the negative. (Cf. M. SADIQ ZIA, Studies in ring theory, Dissertation, L. Eötvös University, Budapest, 1975.)

Now we prove the following

Theorem 4.4. *Assume that the subrings of a ring A constitute a distributive lattice with respect to addition and intersection. Then a subring of the ring A is an (m, n)-quasi-ideal of A if and only if it is the intersection of an $(m, 0)$-ideal and a $(0, n)$-ideal of A.*

PROOF. Let A be a ring with distributive subring lattice, let B and C be an $(m, 0)$-ideal and a $(0, n)$-ideal of A, respectively. Then we have $B^m A \subseteqq B$ and $AC^n \subseteqq C$, whence

$$(B \cap C)^m A \cap A(B \cap C)^n \subseteqq B \cap C,$$

and since the common part of subrings is again a subring, we obtain that the intersection $B \cap C$ is an (m, n)-quasi-ideal of A.

Conversely, suppose that S is an (m, n)-quasi-ideal of A. Then we have $S^m A \cap AS^n \subseteqq S$. We prove that

$$S = \{S\}_{(m, 0)} \cap \{S\}_{(0, n)}$$

provided that the subring lattice of A is distributive. By the property *e)* of Section 2, we have

$$\{S\}_{(m, 0)} = S + S^m A,$$

and

$$\{S\}_{(0, n)} = S + AS^n.$$

The assumed distributivity of the subring lattice yields

$$(S + S^m A) \cap (S + AS^n) = S + (S^m A \cap AS^n) = S,$$

as we stated.

5. (m, n)-ideals in von Neumann regular rings

Definition 5.1. A ring A is called *regular* (in the sense of J. VON NEUMANN), if to every element a of A there exists an element x in A such that $axa = a$.

Theorem 5.2. *In a von Neumann regular ring A every (m, n)-ideal S is an (m, n)-quasi-ideal of A and conversely.*

PROOF. Let A be a von Neumann regular ring. We show that

$$S^m AS^n = S^m A \cap AS^n$$

if S is an (m, n)-ideal of A. From the proof of Theorem 4.2 we know that $S^m AS^n \subseteqq S^m A \cap AS^n$. Conversely, the regularity criterion of L. KOVÁCS [5] implies

$$S^m A \cap AS^n \subseteqq (S^m A)(AS^n) \subseteqq S^m AS^n.$$

Corollary 5.3. *In a von Neumann regular ring A every bi-ideal of A is a quasi-ideal of A (and conversely).*

For this corollary, see S. LAJOS [7].

In what follows we need the following well known lemma of V. A. ANDRUNAKIEVIČ [1].

Lemma 5.4. *Suppose that B is an ideal of a ring A and C is an ideal of the ring B. Let C^* be the ideal of A generated by C. Then*

$$(C^*)^3 \subseteqq C.$$

A consequence of Lemma 5.4 reads as follows.

Corollary 5.5. *If every two-sided ideal of a ring A is idempotent then every i^k-ideal of A is a two-sided ideal of A.*

PROOF. The assumption and Lemma 5.4 imply

$$(C^*)^3 \subseteqq C \subseteqq C^*$$

and $(C^*)^2 = C^*$, that is, $C = C^*$ holds, which completes the proof of Corollary 5.5.

Corollary 5.6. *In a von Neumann regular ring A every i^k-ideal is a two-sided ideal of A.*

PROOF. Evidently, every one-sided (left or right) ideal of a von Neumann regular ring is idempotent. Thus Corollary 5.5 implies Corollary 5.6. [1])

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