Rings (semigroups) containing minimal (0-minimal) right and left ideals

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In the memory of Professor A. KERTÉSZ

§ 1. Introduction

According to a theorem of A. H. CLIFFORD [2] by a *completely 0-simple semigroup* we shall mean a 0-simple semigroup S containing at least one 0-minimal left ideal and at least one 0-minimal right ideal.

R. P. RICH [9] is concerned with the following question: When does a semigroup S with 0 contain a two-sided ideal I such that it is a completely 0-simple subsemigroup of S? Rich's answers are the following two interesting propositions.

Proposition 1.1. (See R. P. RICH [9].) *Let M be a 0-minimal two-sided ideal of a semigroup S with 0 such that $M^2 \neq 0$. If M contains at least one 0-minimal left ideal of S and at least one 0-minimal right ideal of S, then M is a completely 0-simple subsemigroup of S.*

We shall say that a two-sided ideal M of a semigroup S with 0 is a *completely 0-simple ideal* of S if M is a completely 0-simple subsemigroup of S.

Proposition 1.2. (See R. P. RICH [9].) *Let R and L be 0-minimal right and left ideals of a semigroup S with 0, respectively, such that $RL \neq 0$ and $LR \neq 0$, then LR is a completely 0-simple ideal of S.*

Conversely, if M is a completely 0-simple ideal of S, then $M = LR$, where L and R are 0-minimal left and right ideals of S, respectively, and $RL \neq 0$.

The first purpose of this paper is to prove the following sharpening of a part of Proposition 1.2.

Theorem 1.3. *Let R and L be 0-minimal right and left ideals of a semigroup S with 0, such that $RL \neq 0$. Then:* (i) *$K = SRLS$ is a completely 0-simple ideal of S;* (ii) *for every non-zero element a of RL, $K = SaS$ holds.*

We shall also prove two corollaries of this theorem. (See Corollaries 3.1 and 3.7.)

In order to give an analogue of Theorem 1.3 for rings we need some preliminaries.

Proposition 1.4. (See B. L. van der WAERDEN [13], § 123.) *If L is a non-nilpotent minimal left ideal of a ring A, then there exists an idempotent element e in L such that $L = Ae$.*

Since a simple ring has no non-zero nilpotent ideals, it holds

Proposition 1.5. (Cf. E. ARTIN—C. J. NESBITT—R. M. THRALL [1], Corollary 5.4B.) Let e be a non-zero idempotent element of a simple ring A. Then Ae is a minimal left ideal of A if and only if eA is a minimal right ideal of A.

Propositions 1.4 and 1.5 imply

Corollary 1.6. A simple ring contains at least one minimal left ideal if and only if it contains at least one minimal right ideal.

Thus the completely 0-simple semigroups and the simple rings having at least one minimal right or left ideal can be considered as analogous notions. This analogy is supported also by the structure theorems of D. REES [8] and M. PETRICH [7], Theorem II. 2.8, respectively.

According to these remarks we shall use the following terminology.

By a *completely simple ring* we shall mean a simple ring having at least one minimal left or right ideal.

We shall say that a two-sided ideal M of a ring A is a *completely simple ideal* of A if M is a completely simple subring of A.

Another purpose of this paper is to prove the following analogue of Theorem 1.3.

Theorem 1.7. *Let R and L be minimal right and left ideals of a ring A, respectively, such that $RL\neq 0$. Then:* (i′) *$K=ARLA$ is a completely simple ideal of A;* (ii′) *for every non-zero element x of RL, $K=AxA$ holds.*

In § 3 we shall also prove two corollaries of this theorem. (See Corollaries 3.3 and 3.8.)

In § 4 we shall give some examples in connection with Corollaries 3.1, 3.3, 3.7 and 3.8.

§ 2. The proofs of the main results

PROOF OF THEOREM 1.3. Let I be a two-sided ideal of S such that $R\subseteqq I$ and let x be a non-zero element of L. Then the product Ix is a left ideal of S contained in L. By the 0-minimality of L, either $Ix=0$ or $Ix=L$. If $Ix=0$, then the set X of all the elements x of L with $Ix=0$ is a non-zero left ideal of S contained in L. By the 0-minimality of L, we have $X=L$, that is, $IL=IX=0$. By our assumptions $RL\neq 0$ and $R\subseteqq I$, so this case is impossible. We conclude

$$Ix = L \quad \text{(for every non-zero } x\in L). \tag{2.1}$$

If J is a two-sided ideal of S such that $L\subseteqq J$, then one can show dually

$$yJ = R \quad \text{(for every non-zero } y\in R). \tag{2.2}$$

Since $0\neq RL\subseteqq R\cap L$, from (2.1) and (2.2) it follows that

$$Sa = L,\quad aS = R \quad\text{and}\quad SaS = LS = SR \quad \text{(for every non-zero } a \text{ of } RL). \tag{2.3}$$

Hence

$$\begin{matrix} SRL = L, \quad RLS = R, \quad SaS = SRLS = SaS\cdot SaS, \\ R = RLS \subseteqq LS = SaS, \quad L = SRL \subseteqq SR = SaS \end{matrix} \qquad (0\neq a\in RL). \tag{2.4}$$

These relations show, in particular, that the assertion (ii) is true.

Now we prove that K is a 0-minimal two-sided ideal of S. Let C be a non-zero two-sided ideal of S contained in K and let c be a non-zero element of C. Since $C \subseteq K = SaS$ $(0 \neq a \in RL)$ the element c has the form $c = sat$ for some $s, t \in S$. From $c = sat \neq 0$ and $a \in RL$ it follows that sa is a non-zero element of L and at is a non-zero element of R. These facts and the relations (2.1), (2.2), (2.3) imply $L = Ssa = Sa$ and $R = atS = aS$. Now we have

$$ScS = SsatS = SatS = SaS = K.$$

On the other hand, $ScS \subseteq SCS \subseteq C$. So we conclude $C = K$, that is, $K = SaS$ is a 0-minimal two-sided ideal of S. This fact and the properties (2.4) of K imply, by Proposition 1.1, that K is a completely 0-simple ideal of S, indeed.

Proof of Theorem 1.7. The assertion (ii′) can be proved in the same manner as the assertion (ii) of Theorem 1.3.

In order to prove that the ideal $K = ARLA = AxA$ $(0 \neq x \in RL)$ is a completely simple subring of A, it suffices to show the simplicity of the subring K.

First we remark that the minimality of the ideal $K = AxA$ of A and the relation

$$K = AxA = LA = AR = ARLA = K^2 \quad (0 \neq x \in RL) \tag{2.5}$$

can be shown in the same way as the 0-minimality of the ideal $K = SaS$ of S and the relations (2.3_2), (2.4_1) in the proof of Theorem 1.3.

Now let D be a non-zero two-sided ideal of $K = AxA$. Since $\bar{D} = D + AD + DA + ADA$ is a non-zero ideal of A contained in K, the minimality of K implies that $\bar{D} = K$. This fact and (2.5) imply

$$K = K^3 = K\bar{D}K = K(D + AD + DA + ADA)K \subseteq KDK \subseteq D,$$

whence $D = K$. This means that K is a simple subring of A, indeed.

§ 3. Some corollaries

First we prove the following corollary of Theorem 1.3.

Corollary 3.1. (Cf. R. P. Rich [9], A. H. Clifford—G. B. Preston [3], Theorem 6.25.) *The following conditions on a semigroup S with 0 are equivalent:*

(1) *S has at least one 0-minimal right ideal R and at least one 0-minimal left ideal L such that $(LR)^2 \neq 0$;*

(2) *S has at least one 0-minimal right ideal R and at least one 0-minimal left ideal L such that $LR \neq 0$ and $RL \neq 0$;*

(3) *S has at least one 0-minimal right ideal R and at least one 0-minimal left ideal L such that $RL \neq 0$;*

(4) *S has at least one completely 0-simple ideal.*

Proof. The implications (1)⇒(2) and (2)⇒(3) are trivial.

The implication (3)⇒(4) is valid by Theorem 1.3.

(4)⇒(1). Let K be a completely 0-simple ideal of S. By Proposition 1.2, there exist at least one 0-minimal left ideal L of S and at least one 0-minimal right ideal

R of S such that $K=LR$ and $RL\neq 0$. Since K is a completely 0-simple semigroup, we have $0\neq K^2=(LR)^2$. Q.e.d.

In order to prove a corollary of Theorem 1.7 analogous to Corollary 3.1 we need an analogue of Proposition 1.2.

Corollary 3.2. (Cf. R. P. RICH [9].) *If M is a completely simple ideal of a ring A, then $M=LR$, where L and R are minimal left and right ideals of A, respectively, and $RL\neq 0$.*

PROOF. Let L be a minimal left ideal of the completely simple subring M of A. Since M has no non-zero nilpotent left ideals, from Proposition 1.4 it follows that $L=Me$ $(0\neq e=e^2\in L)$. On the other hand, by Proposition 1.5, $eM=R$ is a minimal right ideal of M. Since M is simple, we have $LR=Me^2M=M$. Furthermore, $RL=eM.Me\neq 0$. Evidently, $L=Me$ and $R=eM$ are minimal left and right ideals of A, respectively.

Corollary 3.3. The following conditions on a ring A are equivalent:

(1′) *A has at least one minimal right ideal R and at least one minimal left ideal L such that $(LR)^2\neq 0$;*

(2′) *A has at least one minimal right ideal R and at least one minimal left ideal L such that $LR\neq 0$ and $RL\neq 0$;*

(3′) *A has at least one minimal right ideal R and at least one minimal left ideal L such that $RL\neq 0$;*

(4′) *A has at least one completely simple ideal.*

The PROOF runs similarly to that of Corollary 3.1, but the implication $(4')\Rightarrow(1')$ can be proved by Corollary 3.2 instead of Proposition 1.2.

In order to obtain another corollary of Theorems 1.3 and 1.7, respectively, we need some known notions and results.

A ring (semigroup) A is called *regular* if every element a of A has the property $a\in aAa$.

Proposition 3.4. (See J. A. GREEN [4] and F. SZÁSZ [12].) *A completely 0-simple semigroup (completely simple ring) is regular.*

Proposition 3.5. (See L. KOVÁCS [6] and K. ISÉKI [5].) *A ring* (semigroup) *A is regular if and only if*

$$RL = R\cap L$$

for every right ideal R and every left ideal L of A.

A non-empty subset (an additive subgroup) Q of a semigroup (ring) A is called a *quasi-ideal* of A if $AQ\cap QA\subseteq Q$.

A non-zero quasi-ideal Q of a ring (semigroup) A (with 0) is called *minimal* (*0-minimal*) if Q does not contain properly any non-zero quasi-ideal of A.

Proposition 3.6. (See O. STEINFELD [10] and [11].) *The intersection $R\cap L$ of a minimal (0-minimal) right ideal R and a minimal (0-minimal) left ideal L of a ring (semigroup) A (with 0) is either 0 or a minimal (0-minimal) quasi-ideal of A.*

Now we are going to prove another corollary of Theorem 1.3.

Corollary 3.7. If the product RL of a 0-minimal right ideal R and a 0-minimal left ideal L of a semigroup S with 0 is not 0, then $RL=R\cap L$ and RL is a 0-minimal quasi-ideal of S.

PROOF. By Theorem 1.3 the assumption $RL \neq 0$ implies that $K = SRLS$ is a completely 0-simple subsemigroup of S and R, L are 0-minimal right and 0-minimal left ideals of K, respectively. From Propositions 3.4, 3.5 and 3.6 it follows that $RL = R \cap L$ and RL is a 0-minimal quasi-ideal of S, indeed.

Similarly one can prove the analogous

Corollary 3.8. If the product RL of a minimal right ideal R and a minimal left ideal L of a ring A is not 0, *then* $RL = R \cap L$ *and RL is a minimal quasi-ideal of A.*

REMARK. This result was proved in the paper O. STEINFELD [10] by a direct method.

§ 4. Some remarks and examples

REMARK 4.1. The following example shows that there exists a semigroup S with 0 with the following properties: S has 0-minimal left ideals L_1, L_2 and 0-minimal right ideals R_1, R_2 such that $R_1 L_1 \neq 0$, $L_1 R_1 = 0$ and $R_2 L_2 \neq 0$, $L_2 R_2 \neq 0$.

EXAMPLE 4.1. Let $S = (0, a, l, r, s)$ be a semigroup with the Cayley-table:

	0	a	l	r	s
0	0	0	0	0	0
a	0	0	a	0	r
l	0	0	l	0	s
r	0	a	a	r	r
s	0	l	l	s	s

Evidently, $L_1 = (0, a, l)$, $L_2 = (0, r, s)$ are 0-minimal left ideals of S and $R_1 = (0, a, r)$, $R_2 = (0, l, s)$ are 0-minimal right ideals of S such that $R_1 L_1 = (0, a)$, $L_1 R_1 = 0$ and $R_2 L_2 = (0, s)$, $L_2 R_2 = S$. By Theorem 1.3, $SR_1 L_1 S = SaS = S = SsS = SR_2 L_2 S$ is a completely 0-simple ideal of S, that is, S is a completely 0-simple semigroup.

REMARK 4.2. Let S be a semigroup with 0. From Corollaries 3.1 and 3.7 it follows: if S contains at least one completely 0-simple ideal, then S has the following properties

(5) S has at least one 0-minimal left ideal L and at least one 0-minimal right ideal R such that $LR \neq 0$;

(6) S has at least one 0-minimal quasi-ideal;

(7) S has at least one 0-minimal left ideal L and at least one 0-minimal right ideal R such that $L^2 = L$ and $R^2 = R$.

The following example shows that the conditions (5), (6) and (7) together do not imply condition (4) of Corollary 3.1.

EXAMPLE 4.2. Let $S = (0, a, b, c)$ be a semigroup with the Cayley-table:

	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	b
c	0	0	0	c

Evidently, $L_1=(0, a)$ and $R=(0, c)$ are 0-minimal left and 0-minimal right ideals of S, respectively, such that $L_1^2=L_1$ and $R^2=R$. Furthermore, L_1 and R are 0-minimal quasi-ideals of S, too.

On the other hand, $L_2=(0, b)$ is a 0-minimal left ideal of S such that $L_2R\neq 0$.

Since S has no completely 0-simple ideals, conditions (5), (6) and (7) do not imply condition (4) of Corollary 3.1.

REMARK 4.3. Let A be a ring containing at least one completely 0-simple ideal. Corollaries 3.3 and 3.7 imply that A has the following properties:

(5′) A has at least one minimal left ideal L and at least one minimal right ideal R such that $LR\neq 0$;

(6′) A has at least one minimal quasi-ideal;

(7′) A has at least one minimal left ideal L and at least one minimal right ideal R such that $L^2=L$ and $R^2=R$.

The following example shows that these three conditions together do not imply condition (4′) of Corollary 3.3.

EXAMPLE 4.3. Let D be a division ring and let T denote the ring of all triangular matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \quad (a_{11},\ a_{12},\ a_{22}\in D).$$

Consider the matrices

$$E_{11} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$$

where e denotes the identity element of D. Evidently, TE_{11}, TE_{12} are minimal left ideals of T, $E_{22}T$ is a minimal right ideal of T and $E_{11}TE_{11}$ is a minimal quasi-ideal of T such that

$$(TE_{11})^2 = TE_{11}, \quad (E_{22}T)^2 = E_{22}T, \quad TE_{11}\cdot E_{12}T \neq 0.$$

These facts mean that T has the properties (5′), (6′) and (7′), but evidently T has no completely simple ideals.

References

[1] E. ARTIN—C. J. NESBITT—R. M. THRALL, Rings with minimum condition, *Ann. Arbor,* 1944.
[2] A. H. CLIFFORD, Semigroups without nilpotent ideals, *Amer. J. Math.* **71** (1949), 834—844.
[3] A. H. CLIFFORD—G. B. PRESTON, *The algebraic theory of semigroups, II.* Providence, 1967.
[4] J. A. GREEN, On the structure of semigroups, *Ann. of Math.* **54** (1951), 163—172.
[5] K. ISÉKI, A characterization of regular semigroups, *Proc. Japan Acad.* **32** (1956), 676—677.
[6] L. KOVÁCS, A note on regular rings, *Publ. Math. (Debrecen)* **4** (1956), 465—468.
[7] M. PETRICH, Rings and semigroups, *Berlin—Heidelberg—New York,* 1974.
[8] D. REES, On semi-groups, *Proc. Cambridge Phil. Soc.* **36** (1940), 387—400.
[9] R. P. RICH, Completely simple ideals of a semigroup, *Amer. J. Math.* **71** (1949), 883—885.
[10] O. STEINFELD, Über die Quasiideale von Ringen, *Acta Sci. Math.* **17** (1956), 170—180.
[11] O. STEINFELND, Über die Quasiideale von Halbgruppen mit eigentlichem Suschkewitsch-Kern, *Acta Sci. Math.* **18** (1957), 235—242.
[12] F. SZÁSZ, Über Ringe mit Minimalbedingung für Hauptrechtsideale, II. *Acta Math. Acad. Sci. Hung.* **12** (1961), 417—439.
[13] B. L. van der WAERDEN, Moderne Algebra, II. Berlin, 1955.

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