Notes on matrix-valued stationary stochastic processes I

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1. Spectral representation

1.0. The spectral representation of the matrix-valued stationary stochastic process (m. st. p.) is given in this study. The m. st. p. — in quadratic case — is defined by T. BALOGH in [1]. Here the "matrix-valued random variable" denotes a $p\times q$-type matrix the elements of which are complex random variables. The values of

$$p \quad \text{and} \quad q \quad \text{are} \quad 1, 2, 3, \dots (<\infty); \quad 1, 2, 3, \dots$$

respectively. The discrete m. st. p. will be examined when the values of p and q are arbitrarily chosen constants. Specifically it means

— a p-dimension vector $(p\geqq 1;\ q=1)$

— a $p\times p$-type quadratic matrix $(p=q)$

— a $p\times q$-type rectangle matrix $(p\neq q;\ q<\infty)$

— a ribbon-matrix $(p\geqq 1;\ q=+\infty)$ valued discrete process.

1.1. M_p denotes the set of the $p\times p$-type quadratic matrices with complex elements.

$\mathcal{M}_2^{p\times q}$ denotes the set of the $p\times q$-type matrix-valued random variables, the elements of which have got variance, if $q=\infty$ the (1.1) is required too.
If $\xi\in\mathcal{M}_2^{p\times q}$ then $E\xi$ denotes the expectation matrix, that is the matrix of the expectations of the elements. For every $\xi, \eta\in\mathcal{M}_2^{p\times q}$ there are their covariance matrix

$$\operatorname{cov}(\xi, \eta) = \langle \xi - E\xi,\ \eta - E\eta\rangle = E(\xi - E\xi)\{\eta - E\eta)^* \in M_p$$

($*$ denotes, as usual the conjugate of the transpose of matrix.) The

$$\{\xi_n \mid \xi_n\in\mathcal{M}_2^{p\times q},\ E\xi_n = 0,\ n = 0, \pm 1, \pm 2, \dots\} = \{\xi_n\}_{-\infty}^{+\infty}$$

is m. st. p. if

$$\operatorname{cov}(\xi_n, \xi_m) = \Gamma(n-m)$$

for every n, m.
The $\{\xi_n\}_{-\infty}^{\infty}$ m. st. p. is a curve in the $\mathcal{M}_2^{p\times q}$ Q.H. space. The concept of Q.H. — Quasi Hilbert — space is given by B. GYIRES in [2].
(The inner product is in $\mathcal{M}_1^{p\times q}$: $\langle\xi, \eta\rangle = E\xi\eta^*$).
For $\xi\in\mathcal{M}_2^{p\times q}$ the $\underline{\xi}^i$ denotes the vector-valued random variable determined by the i-th row of the matrix, thus $\xi = [\underline{\xi}^i]_{i=1}^p$.

If $q=\infty$ then $\underline{\xi}^i$ is an infinite-dimensional row vector, and the

$$E\underline{\xi}^i\underline{\xi}^{i*} < \infty \tag{1.1}$$

condition is required.

1.2. Let us denote the Q. H. space generated by m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$ and the Hilbert space generated by row vectors of this process with

$$Q_\infty = Q(\{\xi_n\}_{-\infty}^{\infty}), \quad v_\infty = v(\{\underline{\xi}_n^i(i=1, 2, \ldots, p\})_{-\infty}^{\infty})$$

respectively.

1.1. Lemma. *The elements of the Q_∞ Q. H. space are built by exactly those row vectors which are the elements of the v_∞ Hilbert space.*

The state of 1.1. lemma follows obviously from the constructions of Q_∞ and v_∞ spaces.

Now we can define an U unitary operator on the $\{\underline{\xi}_n^i, i=1, 2, \ldots, p\}_{-\infty}^{\infty}$ subset of v_∞,

$$U\underline{\xi}_n^i = \underline{\xi}_{n+1}^i.$$

This U unitary operator based on 4.1. lemma of [3]. (p. 14) can be extended over v_∞ and it will be unitary, too.

After that let V by definition

$$V\xi = [U\underline{\xi}^i]_{i=1}^p, \quad \xi \in Q_\infty$$

an unitary operator on Q_∞ Q. H. space.

Properties of V are:

a) $V\xi_n = V^{n+1}\xi_0 = \xi_{n+1}$

b) $V = [U] = \left[\int_0^{2\pi} e^{i\lambda}\, d\underline{F}(\lambda)\right]$

(see 109 theorem of [4]).

The $\{\xi_n\}_{-\infty}^{\infty}$ m. st. p. can be represented by

$$\xi_n = V^n\xi_0 = \left[\int_0^{2\pi} e^{in\lambda}\, d\big(E(\lambda)\underline{\xi}_0^j\big)\right]_{j=1}^p = \int_0^{2\pi} e^{in\lambda}\, dZ(\lambda) \tag{1.2}$$

where $Z(\lambda)=[E(\lambda)\underline{\xi}_0^j]_{j=1}^p \in \mathcal{M}_2^{p\times q}$ is an orthogonal stochastical measure defined on the interval $[0, 2\pi]$ having the properties

a) σ-additivity

b) for $0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 \leqq 2\pi$

$$\langle Z(\lambda_2)-Z(\lambda_1), Z(\lambda_4)-Z(\lambda_3)\rangle = 0(\in M_p).$$

After (1.2) spectral representation of the m. st. p. we can construct the Herglotz-representation of the $\Gamma(\cdot)$ covariance function:

$$\Gamma(n) = \langle \xi_k, \xi_{k+n}\rangle = \int_0^{2\pi} e^{in\lambda}\, dF(\lambda) \tag{1.3}$$

where $F(\lambda)=\langle Z(\lambda), Z(\lambda)\rangle$ is a matrix-valued function uniquely determined, non-negative definite, Hermitian-simmetric, monotonically non-decreasing, of bounded variation in its elements and defined on the interval $[0, 2\pi]$ reffered to as spectral distribution function of the m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$

Bibliography

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(Received Ferbuary 1, 1976.)