Some extended Hermite—Fejér interpolation processes and their convergence

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1. P. SZÁSZ [3] introduced an interpolation formula what he calls extended Hermite—Fejér interpolation. According to him for given $n+m$ distinct points

$$x_1, x_2, \dots, x_n; \quad \xi_1, \xi_2, \dots, \xi_m, \tag{1.1}$$

there exists a unique polynomial $S(x)$ of degree $\leqq 2n+m-1$ satisfying the following conditions:

$$\begin{aligned} &S(x_k) = f(x_k), \quad S'(x_k) = 0, \quad k = 1, 2, \dots, n, \\ &S(\xi_j) = f(\xi_j), \qquad\qquad\qquad j = 1, 2, \dots, m. \end{aligned} \tag{1.2}$$

SZÁSZ himself constructed interpolatory polynomials when $m=1$, $\xi_1=1$ and x_k's are the zeros of $P_n(x)$, nth Legendre polynomial or $P_n^{(1/2,\,-1/2)}(x)$, nth Jacobi polynomial and proved the convergence for any continuous function $f(x)$.

Further SAXENA [4] has investigated the convergence in the case, when $m=1$, $\xi_1=-1$ and x_k's are the zeros of $P_n^{(-1/2,\,1/2)}(x)$, nth Jacobi polynomial.

In 1965 MERLI [2] constructed polynomials when $m=1$, $\xi_1=0$ and x_k's are zeros of $T_n(x)$, nth Tchebycheff polynomial of the first kind, n even, and proved the convergence to $f(x)$, which is given by

$$f(x) = c + x^2 \Phi(x)$$

where C is any constant and $\Phi(x)$ is continuous function in $[-1, 1]$. Later FONTANELLA [1] proved the Merli's theorem for the function $f(x)$, where $f(x) = c + |x|^\alpha \Phi(x)$, $\alpha > 0$.

The case when $m=2$, $\xi_1=1$ & $\xi_2=-1$ has also been studied by SZÁSZ [5]. Now it is interesting to consider the following cases:

i) When $m=2$, $\xi_1=1$ and $\xi_2=0$;

ii) When $m=2$, $\xi_1=-1$ and $\xi_2=0$;

and

iii) When $m=3$, $\xi_1=1$, $\xi_2=-1$ and $\xi_3=0$.

2. Case i) When $m=2$, $\xi_1=1$ and $\xi_2=0$.

The interpolation polynomials $R_n(f, x)$ of degree $\leqq 2n+1$ satisfying the properties

$$\begin{cases} R_n(f, 1) = f(1) & R_n(f, 0) = 0 \\ R_n(f, x_k) = f(x_k) & R_n'(f, x_k) = 0, \quad k = 1, 2, \dots, n \end{cases} \tag{2.1}$$

is given by

$$(2.2)\qquad R_n(f,x) = f(1)\frac{W_n^2(x)}{W_n^2(1)}x + \sum_{k=1}^{n} f(x_k)\frac{x(1-x)}{x_k(1-x_k)}\left[1-\left\{\frac{1-2x_k}{x_k(1-x_k)}+\frac{W_n''(x_k)}{W_n'(x_k)}\right\}(x-x_k)\right] l_k^2(x),$$

where

$$(2.3)\qquad \begin{cases} W_n(x) = \prod_{k=1}^{n}(x-x_k), \\ l_k(x) = \dfrac{W_n(x)}{W_n'(x_k)(x-x_k)}, \quad k = 1, 2, \dots, n. \end{cases}$$

Theorem 1. *Let* $f(x)=c+|x|^\alpha\Phi(x)$, *where* c *is any constant,* $\alpha\geqq 2$, $\Phi(x)$ *is continuous function in* $[-1,1]$ *and* x_k*'s are the zeros of* $P_n^{(1/2,-1/2)}(x)$, nth *Jacobi polynomial. Then the sequence of polynomials* $\{R_n(f,x)\}$ *given by* (2.2), *converges uniformly to* $f(x)$ *in* $[-1,1]$.

The polynomial $P_n^{(1/2,-1/2)}(x)$ is identical with

$$(2.4)\qquad W_n(x) = \text{const.}\left[\frac{\sin(2n+1)\Theta/2}{\sin\Theta/2}\right]_{x=\cos\Theta},$$

having the zeros

$$x_k = \cos\frac{2k\Pi}{2n+1}, \quad k = 1, 2, \dots, n.$$

This polynomial $W_n(x)$ satisfies the differential equation

$$(1-x^2)W_n''(x)-(1+2x)W_n'(x)+n(n+1)W_n(x) = 0,$$

which reduces (2.2) as

$$R_n(f,x) = f(1)\frac{W_n^2(x)}{(2n+1)^2}x + \sum_{k=1}^{n} f(x_k)\frac{x(1-x)(2x_k-x_k^3-x)}{x_k^2(1-x_k^2)(1-x_k)} l_k^2(x).$$

In particular for[1]) $f(x)=|x|^\alpha\Phi(x)$, $\alpha\geqq 2$, we have

$$(2.5)\qquad R_n(f,x) = \Phi(1)\frac{W_n^2(x)}{(2n+1)^2}x + \sum_{k=1}^{n}\Phi(x_k)|x_k|^{\alpha-2}\frac{x(1-x)(2x_k-x_k^3-x)}{(1+x_k)(1-x_k)} l_k^2(x).$$

Now a quasi Hermite—Fejér interpolation process of degree $\leqq 2n$, satisfying the conditions:

$$R_n^*(f,1) = f(1), \quad R_n^*(f,x_k) = f(x_k), \quad R_n^{*\prime}(f,x_k) = 0 \qquad k = 1, 2, \dots, n,$$

is given by

$$(2.6)\qquad R_n^*(f,x) = f(1)\frac{W_n^2(x)}{W_n^2(1)} + \sum_{k=1}^{n} f(x_k)\frac{1-x}{1-x_k}\left[1+\left\{\frac{1}{1-x_k}-\frac{W_n''(x_k)}{W_n'(x_k)}\right\}(x-x_n)\right] l_k^4(x).$$

[1]) Without no loss of generality, now we will always take $C=0$.

Using (2.4) and $f(x)=|x|^\alpha \Phi(x)$, $\alpha \geqq 2$ we have at once

$$(2.7)\qquad R_n^*(f, x) = \Phi(1)\frac{W_n^2(x)}{(2n+1)^2}+\sum_{k=1}^{n}\Phi(x_k)\frac{1-x}{1-x_k}|x_k|^\alpha\left(\frac{1-xx_k}{1-x_k^2}\right)l_k^2(x).$$

Subtracting (2.5) from (2.7), we obtain

$$R_n^*(f, x)-R_n(f, x) = \Phi(1)\frac{(1-x)W_n^2(x)}{(2n+1)^2}+\sum_{k=1}^{n}\Phi(x_k)\frac{|x_k|^{\alpha-2}(1-x)}{(1-x_k)^2(1+x_k)W_n^2(x_k)}.$$

Since $0\leqq(1-x)W_n^2(x)\leqq 2$ [2]) and $|\Phi(x)|\leqq M$ in $[-1, 1]$, it follows that for $\alpha\geqq 2$

$$|R_n^*(f, x)-R_n(f, x)| < \frac{2M}{(2n+1)^2}+2M\sum_{k=1}^{n}\frac{1}{(1-x_k^2)(1+x_k)W_n^2(x_k)}.$$

But[3])

$$\sum_{k=1}^{n}\frac{1}{(1-x_k^2)(1-x_k)W_n'^2(x_k)} = \frac{2n}{(2n+1)^2}$$

therefore,

$$(2.8)\qquad \lim_{n\to\infty} R_n(f, x) = \lim_{n\to\infty} R_n^*(f, x).$$

Further using the well known result of Szász [3] we have

$$(2.9)\qquad \lim_{n\to\infty} R_n^*(f, x) = f(x), \quad -1\leqq x\leqq 1$$

for $f(x)=|x|^\alpha\Phi(x)$, continuous in $[-1, 1]$ [4]). Combining (2.8) and (2.9) we have the theorem 1.

Theorem 2. *Let* $f(x)=c+|x|^\alpha\Phi(x)$ *where C is any constant and* $\Phi(x)$ *continuous in* $[-1, 1]$. *Then for the sequence of polynomials* (2.2) *constructed on the zeros of* $P_n(x)$, *nth Legendre polynomial, the relation*

$$R_n(f, x)\to f(x) \quad \textit{for} \quad -1<x\leqq 1$$

holds.

The proof of Theorem 2 runs exactly on the same lines as the proof of theorem 1. So we omit the details.

3. Case ii) when $m=2$, $\xi_1=-1$ and $\xi_2=0$.

The interpolation polynomial $H_n(f, x)$ of degree $\leqq 2n+1$ with the properties:

$$H_n(f, -1)=f(-1) \quad H_n(f, 0)=0$$
$$H_n'(f, x_k)=f(x_k) \quad H_n'(f, x_k)=0, \quad k=1, 2, \dots, n$$

is given by

$$(3.1)\qquad H_n(f, x) =$$
$$=f(-1)\frac{W_n^2(x)}{W_n^2(-1)}x+\sum_{k=1}^{n}f(x_k)\frac{x(1+x)}{x_k(1+x_k)}\left[1-\left\{\frac{1+2x_k}{x_k(1+x_k)}+\frac{W_n''(x_k)}{W_n'(x_n)}\right\}(x-x_k)\right]l_k^2(x)$$

where $W_n(x)$ and $l_k(x)$ are given by (2.3).

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[2]) See Szász [3]
[3]) See Szász [3]
[4]) See for example holds G. Vital, G. Sausone, Moderna teoria delle funzioni di variabile reale. *Bologna* (1952).

Theorem 3. *If $\Phi(x)$ is continuous in $[-1, 1]$ and $f(x)=C+|x|^\alpha \Phi(x)$, $\alpha \geqq 2$ and x_k's denote the zeros of Jacobi polynomial $P_n^{(-1/2, 1/2)}(x)$, then the sequence of polynomials* (3.1) *converges uniformly to $f(x)$ in $[-1, 1]$.*

The polynomial $P_n^{(-1/2, 1/2)}(x)$ is identical with

$$(3.2) \qquad W_n(x) = \text{const} \frac{\cos \frac{2n+1}{2} \Theta}{\cos \Theta/2}, \quad x = \cos \Theta$$

having the zeros

$$x_k = \cos \frac{2k-1}{2n+1} \Pi, \quad k = 1, 2, \ldots, n.$$

The differential equation satisfied by $W_n(x)$ in (3.2) is

$$(1-x^2) W_n''(x) + (1-2x) W_n'(x) + n(n+1) W_n(x) = 0$$

which reduces (3.1) for $+(x)=|x|^\alpha \Phi(x)$ as

(3.3)

$$H_n(f, x) = \Phi(-1) \frac{W_n^2(x)}{-(2n+1)^2} x + \sum_{k=1}^{n} \Phi(x_k) \frac{x(1+x) x_k |x_k|^{\alpha-2}}{(1+x_k)^2 (1-x_k)} (2x_k - x_k^2 - x) l_k^2(x).$$

Now a quasi Hermite—Fejér polynomial satisfying the condition:

$$H_n^*(f, -1) = f(-1), \quad H_n^*(f, x_k) = 0, \quad H_n^{*\prime}(f, x_k) = 0, \quad k = 1, 2, \ldots, n,$$

is given by

$$H_n^*(f, x) = f(-1) \frac{W_n^2(x)}{W_n^2(-1)} + \sum_{k=1}^{n} f(x_k) \frac{1+x}{1+x_k} \left[1 - \left\{\frac{1}{1+x_k} + \frac{W_n''(x_k)}{W_n'(x_k)}\right\} (x - x_k)\right] l_k^2(x).$$

For $W_n(x)$ given by (3.2) and $f(x)=|x|^\alpha \Phi(x)$, $\alpha \geqq 2$ we have

$$(3.4) \qquad H_n^*(f, x) = \Phi(-1) \frac{W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^{n} \Phi(x_k) |x_k|^\alpha \frac{(1+x)(1-x x_k)}{(1-x_k)^2 (1+x_k)} l_k^2(x).$$

Saxena [4] proved that

$$(3.5) \qquad H_n^*(f, x) \to f(x), \quad -1 \leqq x \leqq 1$$

as $n \to \infty$ when $f(x)$ is continuous in $[-1, 1]$.
Equation (3.3) and (3.4) yield

$$H_n^*(f, x) - H_n(f, x) = \Phi(-1) \frac{(1+x) W_n^2(x)}{(2n+1)^2} + \sum_{k=1}^{n} \Phi(x_k) |x_k|^{\alpha-2} \frac{(1+x)}{(1+x_k)(1-x_k^2) W_n'^2(x_k)}.$$

Since $|\Phi(x)| \leqq M$ for $-1 \leqq x \leqq 1$, $\alpha \geqq 2$ and[5])

$$0 < (1+x) W_n^2(x) \leqq 2, \quad \sum_{k=1}^{n} \frac{1}{(1-x_k^2)(1+x_k) W_n'^2(x_k)} = \frac{2n}{(2n+1)^2}.$$

[5]) See Saxena [4].

Hence

$$H_n(f, x) \to H_n^*(f, x) \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Equation (3.5) and (3.6) together completes the proof of theorem 3.

4. Case iii), when $m=3$, $\xi_1=1$, $\xi_2=-1$ and $\xi_3=0$.

The interpolation polynomial $Q_n(f, x)$ of degree $\leqq 2n+2$ with the properties:

$$\begin{gathered} Q_n(f, 1) = f(1), \quad Q_n(f, 0) = 0, \quad Q_n(f, -1) = f(-1) \\ Q_n(f, x_k) = f(x_k), \quad Q_n'(f, x_k) = 0, \quad k = 1, 2, \dots, n, \end{gathered} \tag{4.1}$$

is given by

$$\begin{aligned} Q_n(f, x) &= f(1)\frac{W_n^2(x)}{W_n^2(1)}\cdot\frac{(1+x)x}{2} + f(-1)\frac{W_n^2(x)}{W_n^2(-1)}\cdot\frac{(x-1)x}{2} + \\ &+ \sum_{k=1}^{n} f(x_k)\frac{x(1-x^2)}{x_k(1-x_k^2)}\left[1-\left\{\frac{1}{x_k}-\frac{2x_k}{1-x_k^2}+\frac{W_n''(x_k)}{W_n'(x_k)}\right\}(x-x_k)\right] l_k^2(x), \end{aligned} \tag{4.2}$$

where $l_k(x)$ and $W_n(x)$ are given by (2.3).

Theorem 4. *Let $f(x)=C+|x|^\alpha \Phi(x)$, where C is any constant, $\alpha \geqq 2$, $\Phi(x)$ continuous function in $[-1, 1]$ and x_k's be the zeros of $U_n(x)$, nth Tchebycheff polynomial of second kind, n even. Then the sequence of polynomials (4.2) converges uniformly to $f(x)$ in $[-1, 1]$.*

The differential equation satisfied by $W_n(x)=U_n(x)$ is

$$(1-x^2)U_n''(x)-3xU_n'(x)+n(n+2)U_n(x) = 0$$

from which we at once have

$$\begin{aligned} Q_n(f, x) &= f(1)\frac{U_n^2(x)(1+x)x}{2(n+1)^2} + f(-1)\frac{U_n^2(x)(x-1)x}{2(n+1)^2} + \\ &+ \sum_{k=1}^{n} f(x_k)\frac{x(1-x^2)(2x_k-x_k^3-x)}{x_k^2(1-x_k^2)^2} l_k^2(x). \end{aligned}$$

Taking $f(x)=|x|^\alpha \Phi(x)$, $\alpha \geqq 2$, we have

$$\begin{aligned} Q_n(f, x) &= \Phi(1)\frac{U_n^2(x)}{2(n+1)^2}x(1+x) + \Phi(-1)\frac{U_n^2(x)}{2(n+1)^2}x(x-1) + \\ &+ \sum_{k=1}^{n} \Phi(x_k)|x_k|^{\alpha-2}\frac{x(1-x^2)(2x_k-x_k^3-x)}{(1-x_k^2)} l_k^2(x). \end{aligned} \tag{4.3}$$

Now the quasi-Hermite—Fejér interpolation polynomial $Q_n^*(f, x)$ satisfying the properties

$$\begin{gathered} Q_n^*(f, 1) = f(1), \quad Q_n^*(f, -1) = l(-1) \\ Q_n^*(f, x_k) = f(x_k), \quad Q_n^{*\prime}(f, x_k) = 0, \quad k = 1, 2, \dots, n, \end{gathered}$$

is given by

$$\begin{aligned} Q_n^*(f, x) &= f(1)\frac{1+x}{2}\frac{W_n^2(x)}{W_n^2(1)} + f(-1)\frac{1-x}{2}\frac{W_n^2(x)}{W_n^2(-1)} + \\ &+ \sum_{k=1}^{n} f(x_k)\frac{1-x^2}{1-x_k 2}\left[1+\left\{\frac{2x_k}{1-x_k 2}-\frac{W_n''(x_k)}{W_n'(x_k)}\right\}(x-x_k)\right] l_k^2(x). \end{aligned}$$

Which for $W_n(x)=U_n(x)$ & $f(x)=|x|^\alpha \Phi(x)$, $\alpha \geqq 2$ takes the from

$$Q_n^*(f,x) = \Phi(1)\frac{1+x}{2}\frac{U_n^2(x)}{(n+1)^2}+\Phi(-1)\frac{1-x}{2}\frac{U_n^2(x)}{(n+1)^2}+ \tag{4.4}$$
$$+\sum_{k=1}^{n}\Phi(x_k)|x_k|^\alpha\frac{(1-x)^2(1-xx_k)}{(1-x_k^2)^2}l_k^2(x).$$

From (4.3) and (4.4), we get

$$Q_n^*(f,x)-Q_n(f,x) = [\Phi(1)-\Phi(-1)]\frac{1-x^2}{2(n+1)^2}U_n^2(x)+\sum_{k=1}^{n}\Phi(x_k)|x_k|^{\alpha-2}\frac{(1-x^2)U_n^2(x)}{(n+1)^2}.$$

Further we have

$$|Q_n^*(f,x)-Q_n(f,x)| < |\Phi(1)-\Phi(-1)|\frac{(1-x^2)U_n^2(x)}{2(n+1)^2}+\sum_{k=1}^{n}|\Phi(x_k)|\,|x_k|^{\alpha-2}\frac{(1-x^2)U_n^2(x)}{(n+1)^2}.$$

Since $|\Phi(x)|\leqq M$, $\alpha\geqq 2$, $0<(1-x^2)U_n^2(x)<1$ for $-1\leqq x\leqq 1$, therefore

$$\lim_{n\to\infty} Q_n(f,x) = \lim_{n\to\infty} Q_n^*(f,x). \tag{4.5}$$

But theorem of SzÁsz [5] gives

$$Q_n^*(f,x) \to f(x) \quad \text{as} \quad n\to\infty. \tag{4.6}$$

From (4.5) and (4.6) we have the theorem 4.

Theorem 5. *If $f(x)=C+|x|^\alpha\Phi(x)$ where C is any constant, $\alpha\geqq 2$ and $\Phi(x)$ is continuous in $[-1, 1]$, then the sequence of polynomials* (4.2) *constructed on the zeros of $P_n(x)$, nth Legendre polynomial converges uniformly to $f(x)$ in $[-1, 1]$.*

The proof of this theorem runs exactly on the same lines as the proof of the theorem 5. So we omit the details.

Acknowledgement: I am grateful to Dr. K. K. Mathur for his help and guidance.

References

[1] F. Fontanela, Su una formula di interpolazione di tipo misto di Lagrange—Hermite. *Riv. Mat. Univ. Parma* (2) **7** (1966), 151—156.

[2] L. Merli, Le formuledi interpolazione de tipo misto di Lagrange e Hermite perla classe delle funzione del lipo $f(x)=C+x^2\Phi(x)$. *Riv. Mat. Univ. Parma* (2) **6** (1965), 17—21.

[3] P. SzÁsz, The extended Hermite—Fejér interpolation formula with application to the theory of generalised almost step parabolas, Publ. Math. (Debrecen) **11** (1964), 85—100.

[4] R. B. Saxena, On the convergence and divergence behaviour of Hermite—Fejér and extended Hermite—Fejer interpolations. *Rendiconti del Seminario Matematico (Univ. e Politec. di Torino)* **27** (1967—68), 228—238.

[5] P. SzÁsz, On quasi-Hermite—Fejér interpolation, *Acta. Math. Acad. Sci. Hungaricae.* **10** (1959), 413—439.

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LUCKNOW UNIVERSITY, LUCKNOW (INDIA).

(Received February 16, 1976.)