Note on imbedding theorems

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Introduction

Let $\varphi(x) \equiv \varphi_p(x)$, $(p \geqq 1)$ be a nonnegative increasing function on $[0, \infty)$ with the following properties:

$$\frac{\varphi(x)}{x} \uparrow \quad \text{and} \quad \frac{\varphi(x)}{x^p} \downarrow \quad \text{as} \quad x \to \infty. \tag{1}$$

The set of the measurable functions $f(x)$ on (a, b) $(0 \leqq a < b \leqq \infty)$ for which $\int_a^b \varphi(|f(x)|)\, dx < \infty$ will be denoted by $\varphi(L(a, b))$

If $f(x) \in \varphi(L(a, b))$ then the "modulus of continuity of $f(x)$ with respect to φ" will be defined by

$$\omega_\varphi(\delta; f) \equiv \omega_\varphi(\delta; f;\, a, b) = \sup_{0 \leqq h \leqq \delta} \bar{\varphi}\Big(\int_a^{b-h} (|f(x+h) - f(x)|)\Big)\, dx,$$

where $\bar{\varphi}(x)$ denotes *the inverse function of* $\varphi(x)$.

If $\varphi(x) = x^p$ $(p \geqq 1)$ then $\varphi(L(a, b))$ and $\omega_\varphi(\delta; f)$ will be denoted, as usually, by $L^p(a, b)$ and $\omega_p(\delta; f)$.

If given a function $\varphi(x)$ and a nondecreasing continuous function $\omega(x)$ with $\omega(0) = 0$, then $H_\varphi^\omega \equiv H_\varphi^{\omega(\delta)}$ will denote the collection of the functions $f(x)$ satisfying the condition $\omega_\varphi(\delta; f) = O(\omega(\delta))$.

P. L. UL'JANOV proved imbedding theorems in several papers (see for instance [6] and [7]) Among others he gave condition which assure that a function $f(x) \in L^p(0, 1)$ should belong to another space $L^\nu(0, 1)$ if $\nu > p$.

L. LEINDLER generalized these results of Ul'janov in [2] and [3], where he gave, among others, conditions which assure the transition from an arbitrary collection $\varphi_p(L(0, 1))$ to $\varphi_p(L(0, 1))\Lambda(L(0, 1))$ and $\varphi_\nu(L(0, 1))$ where $\Lambda(x)$ is a "slowly increasing" function.

He e.g. proved the following:

Theorem A. ([3], Theorem 1.). *Let* $f(x)\in\varphi\bigl(L(0,1)\bigr)$, $\bigl(\varphi(x)\equiv\varphi_p(x),\ p\geqq 1\bigr)$ *and let* $\{\lambda_k\}$ *be a nonnegative monotonic sequence of numbers such that*[1])

(1) $$\sum_{k=m}^{\infty}\frac{\lambda_k}{k^{1+\varepsilon}}\leqq K(\lambda)\cdot\frac{\lambda_m}{m^{\varepsilon}}$$

where[2]) $\varepsilon=(4[p+1]+2)^{-1}$, *and let*[3]) $\Lambda(x)=\sum_{k=1}^{x}\frac{\lambda_k}{k}$. *Then*

(2) $$\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\,\varphi\left(\omega_{\varphi}\left(\frac{1}{n};f\right)\right)\quad \textit{implies}\quad f(x)\in\varphi\bigl(L(0,1)\bigr)\Lambda\bigl(L(0,1)\bigr).$$

We gave in [5] for certain functions $\omega(\delta)$ necessary and sufficient conditions for the imbedding

$$H_{\varphi}^{\omega(\delta)}\subset\varphi\bigl(L(0,1)\bigr)\Lambda\bigl(L(0,1)\bigr).$$

G. Gajmnasarov proved similar theorems to those of Ul'janov concerning to interval $[0,\infty)$ instead of $[0,1]$. For example he proved the following:

Theorem B. ([1], Theorem 3.) *Let* $f(x)\in L^p(0,\infty)$, $p\geqq 1$, *and let* $1\leqq p<\nu$. *If*

$$\sum_{n=1}^{\infty}n^{\frac{\nu}{p}-2}\,\omega_p^{\nu}\left(\frac{1}{n};f\right)<\infty,\quad \textit{then}\quad f\in L^{\nu}(0,\infty).$$

In the present paper we prove theorems of similar type to those of L. Leindler ([3]) and to our above mentioned result ([5]) concerning to interval $[0,\infty)$ instead of $[0,1]$; namely we give conditions assuring the transition from $\varphi_p\bigl(L(0,\infty)\bigr)$ to $\varphi_p\bigl(L(0,\infty)\bigr)\Lambda\bigl(L(0,\infty)\bigr)$ and to $\varphi_{\nu}\bigl(L(0,\infty)\bigr)$, furthermore for certain functions $\omega(\delta)$ we give necessary and sufficient condition for

$$H_{\varphi}^{\omega(\delta)}\subset\varphi\bigl(L(0,\infty)\bigr)\Lambda\bigl(L(0,\infty)\bigr).$$

More precisely we prove the following theorems:

Theorem 1. *Let* $f(x)\in\varphi\bigl(L(0,\infty)\bigr)$, $\bigl(\varphi(x)\equiv\varphi_p(x),\,p\geqq 1\bigr)$ *and let* $\{\lambda_k\}$ *be a nonnegative monotonic sequence of numbers such that*

(3) $$\sum_{k=m}^{\infty}\frac{\lambda_k}{k}\leqq K(\lambda)\,\frac{\lambda_m}{m^{\varepsilon}},$$

where $\varepsilon=4([p+1]+2)^{-1}$, *and furthermore let*

$$\Lambda(x)=\sum_{k=1}^{x}\frac{\lambda_k}{k}.$$

[1]) K and K_i denote either absolute constans or constans depending on certain functions and numbers which are not necessary to specify; $K(\alpha;\beta)$ and $K_i(\alpha,\beta,\dots)$ denote positive constans depending only on the indicated parameters.

[2]) $[y]$ denotes the integer part of y.

[3]) $\sum_{a}^{b}$ where a and b are not necessarily integers, means a sum over all integers between a and b.

Then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n} \varphi\left(\omega_{\varphi}\left(\frac{1}{n}; f\right)\right) < \infty$$

implies

(4) $$f(x) \in \varphi\big(L(0, \infty)\big) \Lambda\big(L(0, \infty)\big).$$

Theorem 2. *Set* $\varphi(x) \equiv \varphi_p(x)$ *and* $\psi(x) \equiv \varphi_\nu(x)$ $(p, \nu \geqq 1)$ *and suppose that there are* $k(\varphi, \psi)$ *and* $K(\varphi, \psi)$ *constants for which*

(5) $$\psi(x) \leqq K(\varphi, \psi), \quad \text{if} \quad 0 \leqq x \leqq k(\varphi, \psi).$$

Let $\{\varrho_K\}$ *be a nonnegative nondecreasing sequence of numbers with*

$$\sum_{k=m}^{\infty} \frac{\varrho_k}{k^2} \leqq K(\varrho) \frac{\varrho_m}{m},$$

and denote by $\varrho(x)$ *the continuous function which is linear between n and* $n+1$, *furthermore* $\varrho(n) = \varrho_n$. *Suppose that* $f(x) \in f\big(L(0, \infty)\big)$, *then*

(6) $$\sum_{n=1}^{\infty} \frac{\varrho_n}{n^2} \psi\left(\bar{\varphi}\left(n\varphi\left(\omega_{\varphi}\left(\frac{1}{n}; f\right)\right)\right)\right) < \infty$$

implies

(7) $$f(x) \in \psi\big(L(0, \infty)\big) \varrho\big(L(0, \infty)\big).$$

Theorem 3. *Let* $\omega(\delta)$ *be a given nondecreasing continuous function with* $\omega(0) = 0$, *for which there exists the limit*

(8) $$\lim_{h \to 0} \frac{\omega\left(\frac{h}{2}\right)}{\omega(h)},$$

and let $\{\lambda_k\}$ *be a nonnegative monotonic sequence of numbers satisfying* $\lambda_{k^2} \leqq K\lambda_k$ *for any* k. *Then a necessary and sufficient condition that*

(9) $$H_{\varphi}^{\omega(\delta)} \subset \varphi\big(L(0, \infty)\big) \Lambda\big(L(0, \infty)\big)$$

is that

(10) $$\sum_{n=1}^{\infty} \frac{\lambda_n \varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n} < \infty$$

where $\Lambda(x)$ *means the same as in Theorem 1.*

We remark that condition (5) may not be omitted. (See for example the case $\varrho(x) \equiv 1$, $\psi(x) = x^2$, $\varphi(x) = x^4$ and

$$f(x) = \begin{cases} 0, & \text{if} \quad 0 \leqq x \leqq 2 \\ \frac{1}{\sqrt{x}}, & \text{if} \quad 2 \leqq x). \end{cases}$$

We also remark that in the case $\varphi(x)=x^p$, $\psi(x)=x^\nu$ Theorem 2 includes Theorem B; furthermore in the case $\varphi(x)=x$, $\Lambda(n)=\varphi(n)$ Theorem 1 includes Theorem 1 of G. Gajmnasarov [1].

Furthermore we mention that the proofs of the theorems and lemmas run similarly to those of the old ones which were proved by P. L. Ul'janov, L. Leindler, G. Gajmnasarov and me; therefore we detail only those parts wich differ from the earlier ones.

§ 1. Lemmas

Lemma 1. ([7], Lemma 13). *Let $A(u)$ be a nonnegative nondecreasing function on $[0, \infty)$ such that $A(u^2)\leqq KA(u)$ for any $u\in[0, \infty)$ and let $B(u)$ be a nonnegative function on $[0, 1]$. Then*

$$\int_0^1 B(u)A\big(B(u)\big)\,du < \infty$$

implies

$$\int_0^1 B(u)A\left(\frac{1}{u}\right)du < \infty.$$

Lemma 2. ([1], Lemma 5). *If $f(x)\in L(0, \infty)$ and $F(z)$ is a nonnegative nonincreasing function equidistributed with $|f(x)|$, that is*

$$\operatorname{mes}\{x\colon x\in[0, \infty),\ |f(x)|>y\} = \operatorname{mes}\{z\colon z\in[0, \infty),\ F(z)>y\},$$

then

$$\sup_{\substack{E\subset[0,\infty)\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_0^\alpha F(z)\,dz \quad \text{for any} \quad 0\leqq\alpha\leqq\infty;$$

furthermore if $\alpha<\infty$ and

$$\sup_{\substack{E\subset[0,\infty)\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_{E_0} |f(x)|\,dx,$$

then

$$\sup_{\substack{E\subset[0,\infty)-E_0\\ |E|=\alpha}} \int_E |f(x)|\,dx = \int_\alpha^{2\alpha} F(z)\,dz.$$

Lemma 3. ([3], Lemma 3). *Let $\varphi(x)\equiv\varphi_p(x)$. If $u(x)$ and $v(x)$ are nonnegative measurable functions on the interval I, then we have*

$$\varphi\left(\frac{\int_I u(x)v(x)\,dx}{\int_I u(x)\,dx}\right) \leqq 2^p \frac{\int_I u(x)\,\varphi\big(v(x)\big)\,dx}{\int_I u(x)\,dx}.$$

Lemma 4. ([4], Theorem; Inequality (8)).
If $a_n\geqq 0$ and $\lambda_n>0$, then

$$\sum_{n=1}^\infty \lambda_n\varphi\left(\sum_{i=1}^n a_i\right) \leqq K(\varphi)\sum_{n=1}^\infty \lambda_n\varphi\left(\frac{a_n}{\lambda_n}\sum_{K=n}^\infty \lambda_K\right).$$

Lemma 5. *If* $f(x)\in\varphi(L(0,\infty))$ *and*

$$\varphi_n(t)=n\int_t^{t+\frac{1}{n}} f(u)\,du,\quad 0\leqq t<\infty,\quad n=1,2,\ldots,$$

then

$$\int_0^\infty \varphi(|f(t)-\varphi_n(t)|)\,dt\leqq K(\varphi)\,\varphi\left[\omega_\varphi\left(\frac{1}{n};f\right)\right].$$

Proof *of Lemma 5.*

$$\int_0^\infty \varphi(|f(t)-\varphi_n(t)|)\,dt=\int_0^\infty \varphi\left(\left|n\int_0^{\frac{1}{n}} f(t)\,du-n\int_t^{t+\frac{1}{n}} f(u)\,du\right|\right)dt=$$

$$=\int_0^\infty \varphi\left(n\left|\int_0^{\frac{1}{n}}[f(t)-f(u+t)]\,du\right|\right)dt\leqq\int_0^\infty \varphi\left(n\int_0^{\frac{1}{n}}|f(t)-f(u+t)|\,du\right)dt=I.$$

Next we use Lemma 3. and have

$$I\leqq 2^p\int_0^\infty n\left(\int_0^{\frac{1}{n}}\varphi(|f(t)-f(u+t)|)\,du\right)dt=$$

$$=2^p n\int_0^{\frac{1}{n}}\left(\int_0^\infty \varphi(|f(t)-f(u+t)|)\,dt\right)du\leqq 2^p\varphi\left[\omega_\varphi\left(\frac{1}{n},f\right)\right],$$

which proves the statement of Lemma 5.

Lemma 6. *Let* $f(x)\in\varphi(L(0,\infty))$ *and the functions* $\varphi_n(t)$ *same as in Lemma 5 and*

$$\psi(t)\equiv\psi_n(t)=\psi_{n,k}(t)=n\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} f(u)\,du,\quad k+\frac{\nu}{n}\leqq t<k+\frac{\nu+1}{n},$$

where $\nu=0,1,\ldots,n-1;\ k=0,1,2,\ldots;\ n=1,2,\ldots.$ *Then*

$$\int_0^\infty \varphi(|\varphi_n(t)-\psi(t)|)\,dt\leqq K(\varphi)\,\varphi\left[\omega_\varphi\left(\frac{1}{n};f\right)\right].$$

Proof. An easy calculation gives that

$$\int_0^\infty \varphi(|\varphi_n(t)-\psi(t)|)\,dt = \sum_{k=0}^\infty \int_k^{k+1} \varphi(|\varphi_n(t)-\psi(t)|)\,dt =$$

$$= \sum_{k=0}^\infty \sum_{\nu=0}^{\nu-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left\{\left|n\int_t^{t+\frac{1}{n}} f(u)\,du - n\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} f(u)\,du\right|\right\} dt =$$

$$= \sum_{k=0}^\infty \sum_{\nu=0}^{n-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left\{n\left|\int_{k+\frac{\nu+1}{n}}^{t+\frac{1}{n}} f(u)\,du - \int_{k+\frac{\nu}{n}}^{t} f(u)\,du\right|\right\} dt =$$

$$= \sum_{k=0}^\infty \sum_{\nu=0}^{n-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left\{n\left|\int_{k+\frac{\nu}{n}}^{t} \left[f\left(u+\frac{1}{n}\right)-f(u)\right] du\right|\right\} dt \leqq$$

$$\leqq \sum_{k=0}^\infty \sum_{\nu=0}^{n-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left\{n\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \left|f\left(u+\frac{1}{n}\right)-f(u)\right| du\right\} dt = I.$$

By Lemma 3 we have

$$I \leqq 2^p \sum_{k=0}^\infty \sum_{\nu=0}^{n-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} n\left(\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left(\left|f\left(u+\frac{1}{n}\right)-f(u)\right) du\right) dt =$$

$$= 2^p \sum_{k=0}^\infty \sum_{\nu=0}^{n-1} \int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}} \varphi\left(\left|f\left(u+\frac{1}{n}\right)-f(u)\right|\right) du =$$

$$= 2^p \int_0^\infty \varphi\left(\left|f\left(u+\frac{1}{n}\right)-f(u)\right|\right) du \leqq 2^p \varphi\left(\omega_\varphi\left(\frac{1}{n}; f\right)\right),$$

whence the statement of Lemma 6 follows.

Lemma 7. *Let $f(x)\in\varphi(L(0,\infty))$ and let $\psi(t)$ be the same function as in Lemma 6. Then*

$$\int_0^\infty \varphi(|f(t)-\psi(t)|)\,dt \leqq K(\varphi)\,\varphi\left(\omega_\varphi\left(\frac{1}{n}; f\right)\right).$$

Proof. Applying the Lemma 5 and Lemma 6 and the fact

$$\varphi(a+b) \leqq K(\varphi)(\varphi(a)+\varphi(b)) \quad (a \geqq 0,\ b \geqq 0),$$

we obtain the statement of Lemma 7.

Lemma 8. *If $f(x)\in\varphi(L(0,\infty))$, then*

$$\omega_\varphi\left(\frac{1}{n},f\right)\geqq K(\varphi)\bar{\varphi}\left(\frac{1}{n}\varphi\left(n\Big(\int_0^{\frac{1}{n}}F(z)\,dz-\int_{\frac{1}{n}}^{\frac{2}{n}}F(z)\,dz\Big)\right)\right),$$

where $\bar{\varphi}(x)$ denotes the inverse function of $\varphi(x)$ and $F(z)$ has the same meaning as in Lemma 2.

Proof. Let

$$\alpha(t)\equiv\alpha_{n,k}(t)=n\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}}|f(u)|\,du=a_{k,\nu}\geqq 0,$$

if $t\in\left[k+\frac{\nu}{n}\leqq t<k+\frac{\nu+1}{n}\right]$, $k=0,1,\ldots,\ \nu=0,1,\ldots,n-1;\ n=1,2,\ldots.$

Denote by $b_0\geqq b_1\geqq b_2>\ldots$ the nonincreasing rearrangement of the sequence $\{a_{k,\nu}\}$.
Then

$$\int_0^\infty\varphi\left(\left|\alpha\left(t+\frac{1}{n}\right)-\alpha(t)\right|\right)dt=\sum_{k=0}^\infty\sum_{\nu=0}^{n-1}\int_{k+\frac{\nu}{n}}^{k+\frac{\nu+1}{n}}\varphi\left(\left|\alpha\left(t+\frac{1}{n}\right)-\alpha(t)\right|\right)dt=$$

$$=\sum_{k=0}^\infty\sum_{\nu=0}^{n-1}\varphi(|a_{k,\nu+1}-a_{k,\nu}|\frac{1}{n}\geqq\frac{1}{n}\varphi(b_0-b_1).$$

From this point the proof runs on the same line as in Lemma 6 in [3].

The following three lemmas may be proved similarly as Lemma 7, Lemma 8 and Lemma 9 in [3], so the proofs of these lemmas we omit.

Lemma 9. *If $f(x)\in\varphi(L(0,\infty))$, then*

$$F(2^{-n})\leqq K(\varphi)\Big\{\int_0^\infty\varphi(|f(x)|)\,dx+\sum_{K=1}^{n-1}\bar{\varphi}\big(2^K\varphi(\omega_\varphi(2^{-K};f))\big)\Big\}$$

for any $n\geqq 1$, where $F(z)$ has the same meaning as in Lemma 2.

Lemma 10. *If $f(x)\in\varphi_p(L(0,\infty))$ and $R=2^{2p+1}$, then*

$$\int_0^{\frac{1}{R\cdot n}}\varphi(F(x))\,dx\leqq\int_{\frac{1}{R\cdot n}}^{\frac{1}{n}}\varphi(F(x))\,dx+K(\varphi)\,\varphi\left(\omega_\varphi\left(\frac{1}{n};f\right)\right)$$

for any $n\geqq 1$.

Lemma 11. *If $f(x)\in\varphi(L(0,\infty))$ $(\varphi(x)\equiv\varphi_p(x))$ and $\varepsilon=(4[p+1]+2)^{-1}$, then*

$$\int_0^{\frac{1}{n}}\varphi(F(x))\,dx\leqq\frac{K(\varphi)}{n^\varepsilon}\left\{\sum_{k=1}^n k^{\varepsilon-1}\varphi\left(\omega_\varphi\left(\frac{1}{k};f\right)\right)+\int_0^\infty\varphi(F(x))\,dx\right\}$$

for any $n\geqq 1$.

§ 2. Proofs of the theorems

Since the proof of Theorem 1, applying the modified lemmas, runs similarly to that of Theorem 1 in [3], we omit it.

PROOF *of Theorem 2.* Let $F(x)$ be the same function as in Lemma 2. Since for any nonnegative function $\chi(u)$ on $[0, \infty)$

$$\int_0^\infty \chi(|f(x)|)\,dx = \int_0^\infty \chi(F(x))\,dx$$

(see [8] p. 54), we have

$$\int_0^\infty \psi(|f(x)|)\varrho(|f(x)|)\,dx = \int_0^\infty \psi(F(x))\varrho(F(x))\,dx,$$

furthermore, by (5),

$$\int_1^\infty \psi(F(x))\varrho(F(x))\,dx = \int_1^\infty \frac{\psi(F(x))}{\varphi(F(x))}\varphi(F(x))\varrho(F(x))\,dx \leqq$$

$$\leqq K_1\varrho(F(1))K(\varphi, \psi)\int_1^\infty \varphi(F(x))\,dx \leqq K_2\int_0^\infty \varphi(|f(x)|)\,dx,$$

therefore, in order to prove (7) it is sufficient to show that

$$\int_0^1 \psi(F(x))\varrho(F(x))\,dx < \infty. \tag{2.1}$$

The proof of (2.1) runs on the same line as in Theorem 2 of [3].

PROOF *of Theorem 3.* The sufficiency of (10) has been proved by Theorem 1. The necessity of (10) will be proved indirectly. Suppose that

$$\sum_{n=1}^\infty \frac{\lambda_n\varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n} = \infty, \tag{3.1}$$

but (9) holds.

Then we can construct a function $f_0(x)$ conducing to a contradiction. The construction of this function is similar to that of LEINDLER in [2], made in the case $\varphi(x)=x^p$.

We define $f_0(x)$ as follows:

$$f_0(x) = \begin{cases} \pi_n, & \text{if} \quad x = 3\cdot 2^{-n-2}, \\ 0, & \text{if} \quad x = 0, \quad x\in\left[\frac{1}{2}, \infty\right), \quad x = 2^{-n}, \\ \text{linear on } [2^{-n-1}, 3\cdot 2^{-n-2}], & [3\cdot 2^{-n-2}, 2^{-n}], \end{cases}$$

$(n=1, 2, \ldots)$ where $\varrho_n=\bar{\varphi}\left(2^{n+1}\left(\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)-\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right)\right)$. First we show that $f_0(x)\in H_\varphi^{\omega(\delta)}$. Let

$$h\in(2^{-k-3}, 2^{-k-2}], \quad k\geqq 2. \tag{3.2}$$

Then

$$\int_0^\infty \varphi\left(|f_0(t+h)-f_0(t)|\right)dt=\Big(\int_0^{3h}+\int_{3h}^\infty\Big)\varphi\left(|f_0(t+h)-f_0(t)|\right)dt = I_1+I_2.$$

We have that

$$I_1\leqq K(\varphi)\int_0^{4h}\varphi\left(|f_0(x)|\right)dx\leqq K\int_0^{2^{-K}}\varphi\left(|f_0(x)|\right)dx\leqq$$

$$\leqq K\sum_{n=K}^{\infty}\int_{2^{-n-1}}^{2^{-n}}\varphi\left(|f_0(x)|\right)dx\leqq K_1\sum_{n=K}^{\infty}\varphi(\varrho_n)2^{-n-1}=$$

$$\leqq K_1\sum_{n=K}^{\infty}\left[\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)-\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right]=K_1\varphi\left(\omega\left(\frac{1}{2^K}\right)\right)\leqq K_2\varphi(\omega(h)).$$

Next we prove, that for any k:

$$\sum_{n=0}^{k}2^{-n}\varphi\left(\frac{2^n}{2^k}\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right)\leqq K\varphi\left(\omega\left(\frac{1}{2^k}\right)\right). \tag{3.3}$$

To prove (3.3) we mention that, by (8) and (3.1),

$$\lim_{h\to 0}\frac{\omega\left(\frac{h}{2}\right)}{\omega(h)}=1 \tag{3.4}$$

follows, namely, if $\lim\limits_{h\to 0}\dfrac{\omega\left(\frac{h}{2}\right)}{\omega(h)}<q<1$, then we should have

$$\varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\leqq q\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right.$$

which by $\lambda_{K^2}\leqq K_1\lambda_K$, would imply the contrary of (3.1).

By (3.4) we may assume that there exists a positive number α such that $0<\alpha<1$ and that for any $n>n_0$

$$\omega\left(\frac{1}{2^{n-1}}\right)\leqq\sqrt[p]{2}\alpha\cdot\omega\left(\frac{1}{2^n}\right). \tag{3.5}$$

Hence, by $\varphi(kx)\leqq k^p\varphi(x)$ $(k>1)$, we have

$$\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\leqq 2\alpha^p\varphi\left(\omega\left(\frac{1}{2^n}\right)\right), \tag{3.6}$$

or

$$2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right) \leqq \alpha^p 2^n \varphi\left(\omega\left(\frac{1}{2^n}\right)\right). \tag{3.7}$$

Since for any $k<1$, $\bar{\varphi}(kx) \leqq \sqrt[p]{k}\,\bar{\varphi}(x)$ we have by (3.7)

$$\bar{\varphi}\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right) \leqq \alpha\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right), \tag{3.8}$$

and consequently

$$\frac{2^{n-1}}{2^K}\bar{\varphi}\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right) \leqq \frac{\alpha}{2}\frac{2^n}{2^K}\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right). \tag{3.9}$$

Using the following property of $\varphi(x)$: $\varphi(kx) \leqq k\varphi(x)$ for any $k<1$, we obtain by (3.9)

$$\varphi\left(\frac{2^{n-1}}{2^K}\bar{\varphi}\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\right) \leqq \frac{\alpha}{2}\varphi\left(\frac{2^n}{2^K}\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right).$$

Hence we get

$$2^{-n+1}\varphi\left(\frac{2^{n-1}}{2^K}\bar{\varphi}\left(2^{n-1}\varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right)\right)\right) \leqq \alpha 2^{-n}\varphi\left(\frac{2^n}{2^K}\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right),$$

which implies (3.3), since $0<\alpha<1$. Having (3.3) we can estimate I_2. Since

$$|f_0(t+h)-f_0(t)| \leqq h2^{n+2}(\varrho_n+\varrho_{n-1}),$$

if

$$2^{-n-1} \leqq t \leqq 2^{-n}, \quad (1 \leqq n \leqq k-1),$$

thus

$$I_2 \leqq \int_{2^{-k}}^{2^{-1}} \varphi(|f_0(t+h)-f_0(t)|)\,dt = \sum_{n=1}^{k-1} \varphi(|f_0(t+h)-f_0(t)|)\,dt \leqq$$

$$\leqq K(\varphi)\sum_{n=0}^{k} 2^{-n}\varphi\left(\frac{2^n}{2^k}\varrho_n\right) \leqq K_1(\varphi)\sum_{n=0}^{k} 2^{-n}\varphi\left(\frac{2^n}{2^k}\left(\bar{\varphi}\left(2^n\varphi\left(\omega\left(\frac{1}{2^n}\right)\right)\right)\right)\right) \leqq$$

$$\leqq K_2(\varphi)\varphi\left(\omega\left(\frac{1}{2^k}\right)\right) \leqq K_3(\varphi)\varphi(\omega(h)).$$

Summing up we get

$$f_0(x) \in H_\varphi^\omega.$$

Finally we prove that

$$f_0(x) \notin \varphi(L(0,\infty))\Lambda(L(0,\infty)).$$

By (3.1)

$$\sum_{n=1}^{N} \frac{\lambda_n\varphi\left(\omega\left(\frac{1}{n}\right)\right)}{n} \to \infty \quad \text{as} \quad N\to\infty. \tag{3.10}$$

Using (3.10), and that $\lambda_{2n} \leqq K_1 \lambda_n$, furthermore that for any N there exists an integer N_1 such that

$$\varphi\left(\omega\left(\frac{1}{N_1}\right)\right) \leqq \frac{1}{nK_1} \varphi\left(\omega\left(\frac{1}{N}\right)\right),$$

an easy computation gives that

$$\sum_{n=1}^{\mu} \Lambda(2^n)\varphi(\varrho_n)2^{-n} \to \infty \quad \text{as} \quad \mu \to \infty. \tag{3.11}$$

Indeed, if $2^\mu > N_1$, we have

$$\sum_{k=1}^{N} \lambda_k k^{-1} \varphi\left(\omega\left(\frac{1}{k}\right)\right) \leqq 2 \sum_{k=1}^{N} \lambda_k k^{-1} \left[\varphi\left(\omega\left(\frac{1}{k}\right)\right) - 2K_1 \varphi\left(\omega\left(\frac{1}{N_1}\right)\right) \leqq\right.$$

$$\leqq 2 \sum_{k=1}^{2^\mu} \lambda_K k^{-1} \left[\varphi\left(\omega\left(\frac{1}{k}\right)\right) - 2K_1 \varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] \leqq$$

$$\leqq 2\left[\sum_{n=1}^{\mu} \sum_{k=2^{n-1}+1}^{2^n} \lambda_k k^{-1} \varphi\left(\omega\left(\frac{1}{k}\right)\right) - 2K_1 \Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] + K_2 \leqq$$

$$\leqq 2\left[\sum_{n=1}^{\mu} \varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right) \sum_{k=2^{n-1}+1}^{2^n} \lambda_k k^{-1} - 2K_1 \Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] + K_2 \leqq$$

$$\leqq 2\left[\sum_{n=2}^{\mu} 2K_1 \varphi\left(\omega\left(\frac{1}{2^{n-1}}\right)\right) \sum_{k=2^{n-2}+1}^{2^{n-1}} \lambda_k k^{-1} - 2K_1 \Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] + K_3 \leqq$$

$$\leqq K_4 \left[\sum_{i=1}^{\mu-1} \varphi\left(\omega\left(\frac{1}{2^i}\right)\right) \sum_{k-2^{i-1}+1}^{2^i} \lambda_k k^{-1} - \Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] + K_3 \leqq$$

$$\leqq K_4 \left[\sum_{i=1}^{\mu-1} \varphi\left(\omega\left(\frac{1}{2^i}\right)\right) (\Lambda(2^i) - \Lambda(2^{i-1})) - \Lambda(2^\mu)\varphi\left(\omega\left(\frac{1}{2^\mu}\right)\right)\right] + K_5 \leqq$$

$$\leqq K_4 \sum_{n=1}^{\mu-1} \Lambda(2^n)\left[\varphi\left(\omega\left(\frac{1}{2^n}\right)\right) - \varphi\left(\omega\left(\frac{1}{2^{n+1}}\right)\right)\right] + K_5 \leqq$$

$$\leqq K_4 \sum_{n=1}^{\mu} \Lambda(2^n)\varphi(\varrho_n)2^{-n-1} + K_5,$$

which proves (3.11) by (3.10).
It is clear that for any m

$$\int_{1/2^{m+1}}^{1} \varphi(|f_0(x)|)\Lambda\left(\frac{1}{x}\right)dx = \sum_{n=0}^{m} \int_{2^{-n-1}}^{2^{-n}} \varphi_0(|f(x)|)\Lambda\left(\frac{1}{x}\right)dx \geqq$$

$$\geqq \sum_{n=0}^{m} \Lambda(2^n) \int_{2^{-n-1}}^{2^{-n}} \varphi(|f_0(x)|)\,dx \geqq K_6 \sum_{n=0}^{m} \Lambda(2^n)\varphi(\varrho_n)2^{-n},$$

and thus, by (3.11) we get

$$\int_0^1 \varphi(|f_0(x)|)\Lambda\left(\frac{1}{x}\right)dx = \infty. \tag{3.12}$$

Since $\lambda_{K^2} \leqq K\lambda_K$, we have

$$\Lambda(n^2) \in K_2 \Lambda(n) \tag{3.13}$$

thus, by (3.12), applying Lemma 1, we obtain

$$\int_0^1 \varphi(|f_0(x)|)\Lambda(|f_0(x)|) = \infty. \tag{3.14}$$

Using (3.13) and the properties of the function $\varphi(x)$, we have

$$\Lambda(\varphi(x)) \leqq K_3 \Lambda(x), \tag{3.15}$$

whence by (3.14) and (3.15)

$$\int_0^\infty \varphi(|f_0(x)|)\Lambda(|f_0(x)|) = \int_0^1 \varphi(|f_0(x)|)\Lambda(|f_0(x)|) = \infty$$

follows, that is,

$$f_0(x) \notin \varphi(L(0, \infty))\Lambda(L(0, \infty)).$$

The proof is completed.

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(Received April 6, 1976.)