On a numbertheoretical series

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A composite number n is called a *pseudoprime* if $n \mid 2^n-2$.

Let $P(x)$ denote the number of pseudoprimes $\leqq x$ and P_n the n-th pseudoprime. In 1949 P. ERDŐS stated that

$$c_1 \log x < P(x) < c_2 x/(\log x)^k, \quad \text{for every } k \text{ and } x > x_0(k). \tag{1}$$

K. SZYMICZEK [10] proved, using the following result of P. ERDŐS (see [2])

$$P(x) < 2x \exp\left\{-\frac{1}{3}(\log x)^{1/4}\right\} \quad \text{if} \quad x > x_0 \tag{2}$$

that $1/P_n < 2/n(\log n)^{4/3}$. Therefore $\sum_{n=1}^{\infty} 1/P_n < \sum_{n=1}^{\infty} 2/n(\log n)^{4/3}$ and since the last series is convergent $\sum_{n=1}^{\infty} 1/P_n$ is also convergent. A. ROTKIEWICZ [9] proved that

$$P(x) > \frac{5}{8}\log_2 x \quad \text{for} \quad x \geqq 1905, \quad \text{where} \quad \log_2 x \quad \text{denotes} \tag{3}$$

the logarithm at the base 2. This result is much stronger than the theorem of K. SZYMICZEK [10]:

$$P(x) > \frac{1}{4}\{\log x + \log\log x + \dots + \underbrace{\log\log \dots \log}_{k \text{ times}} x\} \tag{4}$$

ROTKIEWICZ ([7], problem 47) asked whether the series $\sum 1/\log P_n$ is convergent. A. MĄKOWSKI [5] proved that the series $\sum 1/\log P_n(c)$ is divergent, where $P_n(c)$ denotes the n-th pseudoprime with respect to c (n is a pseudoprime with respect to c if n is composite and $n \mid c^n - c$). He used the fact established by CIPOLLA [1] that the number $(c^{2p}-1)/(c^2-1)$ is a pseudoprime with respect to c if p is an odd prime such that $p \nmid c^2-1$ and that the series $\sum 1/p$, where p runs over the primes, is divergent.

First we note that the divergence of $\sum_{n=1}^{\infty} 1/\log P_n$ follows from the estimation $P(x) > c \log x$. Indeed, if we put $x = P_n$ we get

$$P(P_n) > c \log P_n \tag{5}$$

and since $P(P_n)=n$, by (5)

$$\frac{1}{\log P_n} > c/n$$

and the divergence follows at once from the well-known divergence of the series $\sum_{=1} 1/n$.

Theorem 1. *The series* $\sum 1/\log p_n^{(a)}$, *where* $p_n^{(a)}$ *is the n-th pseudoprime of the form* $ax+b$, *where* $(a, b)=1$, *is divergent.*

PROOF. Let a, b be fixed coprime positive integers. Let $P^{(a)}(x)$ denote the number of pseudoprimes $\equiv b \pmod a$ and $\leqq x$. By theorem 5 of A. ROTKIEWICZ [8] we have that

$$P^{(a)}(x) \gg \log x/(a^c \log\log x), \tag{6}$$

where c is an absolute constant.

If we put $x=P_n^{(a)}$ in (6) we get

$$n \gg \log P_n^{(a)}/\log\log P_n^{(a)} \tag{7}$$

and hence $\log\log P_n^{(a)} \ll \log n$. Thus by (7)

$$\log P_n^{(a)} \ll n(\log\log P_n^{(a)}) \ll n \log n. \tag{8}$$

Hence it follows that

$$\sum 1/\log P_n^{(a)} \gg \sum 1/n \log n \tag{9}$$

and the divergence of the first series in (9) follows from the well-known divergence of $\sum 1/n \log n$. This completes the proof of Theorem 1.

Theorem 2. *If* $n \geqq 7$, *then*

$$e^{\log n + c\sqrt{(\log n)(\log\log n)}} < P_n < e^{\frac{8}{5} n \log 2}$$

where c is an absolute positive constant.

PROOF. In 1955 P. ERDŐS [4] proved with Knödel's method (see [6]) that,

$$P(x) < xe^{-c\sqrt{(\log x)(\log\log x)}} \tag{10}$$

where c is an absolute positive constant. Put $x=P_n$, then by (10)

$$P_n > ne^{c\sqrt{(\log P_n)(\log\log P_n)}} > ne^{c\sqrt{(\log n)(\log\log n)}}. \tag{11}$$

This completes the proof of the left side of the inequality. On the other hand by (3) with $x=P_n$, $P_7=1905$,

$$n > \frac{5}{8}\log_2 P_n \quad \text{if} \quad n \geqq 7, \tag{12}$$

and so clearly

(13) $$e^{\frac{8}{5}n\cdot\log 2} > P_n \quad \text{if} \quad n \geqq 7.$$

This completes the proof.

Theorem 3. *The series* $\sum_{n=1}^{\infty} e^{c\sqrt{(\log P_n)(\log\log P_n)}}/P_n(\log n)^s$ *is convergent for* $s>1$ *(c is the constant ocurring in the Erdős result).*

Proof. By (11)

$$P_n > ne^{c\sqrt{(\log P_n)(\log\log P_n)}}$$

hence

$$1/n(\log n)^s > e^{c\sqrt{(\log P_n)(\log\log P_n)}}/P_n(\log n)^s$$

and so the theorem follows from the well-known convergence of the series $\sum_{n=1}^{\infty} 1/(n(\log n)^s)$ for $s>1$. P. Erdős [4] conjectured that

(14) $$P(x) > x^{1-\varepsilon} \quad \text{for every} \quad \varepsilon > 0 \quad \text{and} \quad x > x_0(\varepsilon) \quad \text{and}$$

gave strong reasons for this conjecture.

Theorem 4. *From the conjecture of P. Erdős it follows that the series* $\sum 1/P_n^{1-\varepsilon}$ *is divergent for every* $\varepsilon>0$.

Proof. Suppose that $\varepsilon>0$ then by the conjecture of Erdős:

(15) $$P(x) > x^{1-\varepsilon} \quad \text{for} \quad x > x_0(\varepsilon).$$

It follows that

(16) $$n > P_n^{1-\varepsilon} \quad \text{for} \quad n > n_0(\varepsilon)$$

and hence

(17) $$\sum 1/P_n^{1-\varepsilon} \gg \sum 1/n$$

and it follows that the series $\sum 1/P_n^{1-\varepsilon}$ is divergent. We have seen that the series $\sum 1/\log P_n$ is divergent.

If we could prove that $P(x) \gg (\log x)^k$, then it would follow that the series $\sum 1/(\log P_n)^k$ is divergent. However this result seems hard to get.

Remark. From Theorem 2 it follows that

$$P_n > n^{1+c\sqrt{(\log\log n)/\log n}},$$

where c is an absolute positive constant.

On the other hand from the conjecture of P. Erdős it follows that $P(x) > x^{1-\varepsilon}$ for all $\varepsilon>0$ and all $x > x_0(\varepsilon)$ and so

$$P_n < n^{\frac{1}{1-\varepsilon}} = n^{1+\frac{\varepsilon}{1-\varepsilon}} \quad \text{for} \quad n > n_0(\varepsilon).$$

Thus for all $\bar{\varepsilon}>0$, $P_n<n^{1+\bar{\varepsilon}}$ if $n>n_0(\bar{\varepsilon})$ and hence if Erdős' conjecture is true the following upper and lower bounds of P_n holds provided that n is sufficiently large

$$n^{1+c\sqrt{(\log\log n)/\log n}} < P_n < n^{1+\bar{\varepsilon}}, \tag{18}$$

where c is an absolute positive constant.

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