On a Q-recurrent Finsler space of second order

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1. Introduction

Let Fn be an n-dimensional Finsler space equipped with positively homogeneous metric function $F(x, \dot{x})$ of degree one in its directional arguments and satisfies the requisite conditions [1][1]) imposed upon it. The fundamental metric tensor of the space is given by

(1.1) $$g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x}), \quad (\dot{\partial}_i \equiv \partial/\partial\dot{x}^i).$$

The projective covariant derivative [2] of a vector field $X^i(x, \dot{x})$ with respect to x^k is given by

(1.2) $$x^i_{((k))} = \partial_k x^i - (\dot{\partial}_m x^i) \Pi^m_{rk} \dot{x}^r + x^h \Pi^i_{hk}$$

where the projective connection coefficients

(1.3) $$\Pi^i_{hk}(x, \dot{x}) \overset{\text{def}}{=} G^i_{hk} - \frac{1}{(n+1)} (2\delta^i_{(h} G^r_{k)r} + \dot{x}^i G^r_{rkh})$$

are positively homogeneous of degree zero in $\dot{x}^i$'s and satisfy the following relations:

(1.4) $$\text{a)}\ \Pi^i_{hkr} \dot{x} h = 0 \quad \text{and} \quad \text{b)}\ \dot{\partial}_h \Pi^i_{jk} = \Pi^i_{hjk}.$$

In particular the projective covariant derivative of x^i vanishes i.e.

(1.5) $$\dot{x}^i_{((k))} = 0.$$

The commutation formulae involving the projective covariant derivative of tensor field $T^i_j(x, \dot{x})$ are given by

(1.6) $$\dot{\partial}_h (T^i_{j((k))}) - (\dot{\partial}_h T^i_j)_{((k))} = T^s_j \Pi^i_{shk} - T^i_s \Pi^s_{jhk}$$

and

(1.7) $$2T^i_{j[((h))((k))]} = -(\dot{\partial}_r T^i_j) Q^r_{shk} \dot{x}^s + T^s_j Q^i_{shk} - T^i_s Q^s_{jhk},$$

where the projective entity $Q^i_{hjk}(x, \dot{x})$ is given by

(1.8) $$Q^i_{hjk}(x, \dot{x}) = 2\{\partial_{[k} \Pi^i_{j]h} - (\dot{\partial}_r \Pi^i_{h[j}) \Pi^r_{k]s} \dot{x}^s + \Pi^r_{h[j} \Pi^i_{k]r}\}$$

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[1]) The numbers in square brackets refer to the references given at the end of the paper.

and satisfies the following relation [2]

(1.9) $$Q^i_{hk} = Q^i_{jhk}\dot{x}^j.$$

The first order recurrency condition for the projective entity $Q^i_{hjk}(x, \dot{x})$ is given by

(1.10) $$Q^i_{hjk((l))} = u_l Q^i_{hjk},$$

where $u_l(x)$ is a recurrence vector depending only upon positional coordinates.

Transvecting (1.10) by $\dot{x}^h$ in view of equations (1.5) and (1.9), we get

(1.11) $$Q^i_{jk((l))} = u_l Q^i_{jk}.$$

2. *Q*-recurrent Finsler space of second order

Definition (2.1): In an Fn if the projective Q^i_{hjk} satisfies the relations

(2.1) $$Q^i_{hjk((l))((m))} = b_{lm} Q^i_{hjk}$$

and

(2.2) $$Q^i_{hjk} \neq 0,$$

where $b_{lm}(x, \dot{x})$ is a tensor field depending both upon positional and directional arguments, then it is said to be *Q*-recurrent Finsler space of second order and b_{sm} is called a recurrent tensor field. In this paper we shall denote such a Finsler space by Fn^*.

Theorem (2.1). *A relation between $u_l(x)$ and $b_{lm}(x, \dot{x})$ is given by*

(2.3) $$b_{lm} = u_{l((m))} + u_l u_m.$$

Proof. Differentiating (1.10) projective covariantly with respect to x^m and using equations (1.10) itself and (2.1), we get the required result (2.3).

Theorem (2.2). *In an Fn^* the recurrence tensor field $b_{lm}(x, \dot{x})$ is non-symmetric.*

Proof. Commutating (2.1) with respect to the indices s and m we get

(2.4) $$2Q^i_{hjk[((l))((m))]} = (b_{lm} - b_{ml}) Q^i_{hjk}.$$

In view of the commutation formula (1.7) the above equation reduces to

(2.5) $$\begin{aligned}(b_{lm} - b_{ml}) Q^i_{hjk} = -(\dot{\partial}_r Q^i_{hjk}) Q^r_{lm} + Q^r_{hjk} Q^i_{rlm} - \\ - Q^i_{rjk} Q^r_{hlm} - Q^i_{hrk} Q^r_{jlm} - Q^i_{hjr} Q^r_{klm}\end{aligned}$$

which proves the theorem.

Theorem (2.3). *In an Fn^* if u_s is independent of $\dot{x}^i$ the recurrence tensor field $b_{sm}(x, \dot{x})$ satisfies the following relation*

(2.6) $$\dot{x}^s \{(b_{lm} - b_{ml})_{((s))} - u_s (b_{lm} - b_{ml})\} = 0.$$

PROOF. In view of the commutation formula (1.6) differentiating (2.5) projective covariantly with respect to x^s and using the equations (1.10), (1.11), (2.5) itself and the fact that u_s is independent of $\dot{x}^i$, we have

$$(2.7)\qquad \begin{aligned}(b_{lm}-b_{ml})_{((s))}Q^i_{hjk} &= u_s(b_{lm}-b_{ml})Q^i_{hjk}+\\ &+Q^r_{lm}\{Q^p_{hjk}\Pi^i_{prs}-Q^i_{pjk}\Pi^p_{hrs}-Q^i_{hpk}\Pi^p_{jrs}-Q^i_{hjp}\Pi^p_{krs}\}.\end{aligned}$$

Transvecting (2.7) by $\dot{x}^s$ and noting equation (1.4a), we obtain

$$(2.8)\qquad Q^i_{hjk}\{(b_{lm}-b_{ml})_{((s))}-u_s(b_{lm}-b_{ml})\}\dot{x}^s=0$$

which in view of the assumption that $Q^i_{hjk}\neq 0$, gives the required result.

Theorem (2.4). *In an Fn^* the recurrence tensor field satisfies the following relation:*

$$(2.9)\qquad \dot{x}^s\{(b_{lm}-b_{ml})_{((s))((n))}-u_{s((n))}(b_{lm}-b_{ml})-2(b_{lm}-b_{ml})_{[((n))}u_{s]}\}.$$

PROOF. Differentiating (2.7) projective covariantly with respect to x^n and using (1.10), (1.11), (2.3) and (2.7) itself, we get

$$(2.10)\qquad \begin{aligned}&Q^i_{hjk}[(b_{lm}-b_{ml})_{((s))((n))}-u_{s((n))}(b_{lm}-b_{ml})-2(b_{lm}-b_{ml})_{[((n))}u_{s]}]-\\ &-Q^r_{lm}\{Q^p_{hjk}\Pi^i_{prs((n))}-Q^i_{pjk}\Pi^p_{hrs((n))}-Q^i_{hpk}\Pi^p_{jrs((n))}-Q^i_{hjp}\Pi^p_{krs((n))}\}=0.\end{aligned}$$

Transvecting (2.10) by $\dot{x}^s$ and noting the equations (1.4a) and (1.5), we get

$$(2.11)\qquad Q^i_{hjk}[(b_{lm}-b_{ml})_{((s))((n))}-u_{s((n))}(b_{lm}-b_{ml})-2(b_{lm}-b_{ml})_{[((n))}u_{s]}]\dot{x}^s=0$$

which in view of the assumption (2.2) yields the required result.

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