On measurable solutions of functional equations

By ANTAL JÁRAI (Debrecen)

Introduction

It is well-known that every homomorphizm of the additive group of real numbers into itself, being Lebesgue measurable on a compact set with positive measure is continuous. More generally [see HEWITT and ROSS [11], p. 346] every homomorphizm of a locally compact Hausdorff group into a σ-compact or separable Hausdorff group being left Haar measurable on a compact subset with positive measure is continuous.

Same regularity properties are valid for other functional equations. E.g. in KUREPA's papers [17], [18] the similar results are demonstrated for the functional equations

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

and

$$f(x+y)+f(x-y) = 2f(x)f(y).$$

Results of the same type can be found in HILLE and PHILLIPS' book [13] and IONESCU—TULCEA's paper [14]. In his paper [3] BAKER studies the relation between measurability and continuity for functional equations, in which the domain of the unknown functions is a locally compact Hausdorff group with left Haar measure.

In several cases it is enough to show that the measurable solutions of the functional equation are bounded on a suitable subset and on the basis of this to draw conclusion by means of theorems concerning Lebesgue integral [see ACZÉL [1], ACZÉL and DARÓCZY [2], DARÓCZY [6] and LEE [20]].

This paper's results show that the measurable solutions are bounded and continuous in case of a general type of functional equations.

In the paper we follow FEDERER's terminology [9] concerning measure theory but in the first § the necessary definitions are summed up. Detailed discussion of the facts about topological groups are to be found in HEWITT and ROSS' monography [11].

§ 2. presents the main results. In definition 2.1 we describe the notion of the family of relations uniformly continuous in measure which is related to the concept of absolute continuity of measures. The most general results 2.3, 2.4, 2.5 and 2.6 are based on it. In section 2.7 and 2.8 less general but easily applicable results are given.

In § 3. sufficient conditions are given for the uniform continuity in measure of families of relations. These conditions are connected with the boundedness of the derivate or a Lipschitz type conditions.

In § 4. the main results are applied for some functional equations.

I wish to thank my professor ZOLTÁN DARÓCZY and my colleagues for supporting my work on this paper.

§ 1. Notations and terminology

1.1 Notations. We shall use the notation

$$A \sim B = \{x\colon x \in A \text{ and } x \notin B\}.$$

We shall identify a function or relation with its graph. If f is a relation, dmn f, im f and $f|A$ will denote the domain, the image and the restriction of f to A, respectively. For each class X we let 2^X be the class of all subsets of X. Let $\mathbf{R}$ denote the set of real numbers, and let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. If a and b are extended real numbers, let $]a, b[$, $[a, b]$ and $[a, b[$ denote the suitable open, close and half open intervals, respectively.

1.2 Measures and measurable sets. We say that μ *measures* X, or that μ is a *measure over* X, if and only if X is a set, $\mu\colon 2^X \to [0, \infty]$ and

$$\mu(A) \leqq \sum_{B \in \mathcal{F}} \mu(B) \quad \text{whenever} \quad \mathcal{F} \subset 2^X, \mathcal{F} \text{ countable}, \quad A \subset \bigcup \mathcal{F}.$$

Following Carathéodory, we say that

A is a μ *measurable set* if and only if $A \subset X$ and

$$\mu(T) = \mu(T \cap A) + \mu(T \sim A) \quad \text{whenever} \quad T \subset X.$$

With any measure μ over X, and any set $Y \subset X$, one associates another measure ν over X by the formula

$$\nu(A) = \mu(Y \cap A) \quad \text{for} \quad A \subset X.$$

All μ measurable sets are also ν measurable, and if Y is a μ measurable subset of X, then for every $A \subset X$, $A \cap Y$ is μ measurable if and only if A is ν measurable.

1.3 Radon measures. By a *Radon measure* we mean a measure μ, over a locally compact Hausdorff space X, with the following three properties:

If K is a compact subset of X, then $\mu(K) < \infty$.

If V is an open subset of X, then V is μ measurable and $\mu(V) = \sup\{\mu(K)\colon K$ is compact, $K \subset V\}$.

If A is any subset of X, then

$$\mu(A) = \inf\{\mu(V)\colon V \text{ is open}, \ A \subset V\}.$$

We observe, that, in case μ is a Radon measure over X, Y is μ measurable, $\mu(Y) < \infty$ and $\varepsilon > 0$, there exists a compact subset C of Y, so that $\mu(Y \sim C) < \varepsilon$, moreover, the measure ν over X, defined by the condition $\nu(A) = \mu(A \cap Y)$ for $A \subset X$ is a Radon measure over X.

The most important Radon measures over a locally compact Hausdorff group are the left Haar measures. If G is a locally compact Hausdorff group, then we shall denote an arbitrary but further on fixed left Haar measure over G with λ; if $G=\mathbf{R}^n$, then λ denotes the Lebesgue measure over $\mathbf{R}^n$.

1.4 Measurable functions. Assuming that μ measures X, and Y is a topological space, we say that f is a μ *measurable function* if and only if f is a function whose image is contained in Y, whose domain is contained in X, $\mu(X\sim \operatorname{dmn} f)=0$, and for which $f^{-1}(V)$ is μ measurable whenever V is an open subset of Y.

The following definition are motivated by BOURBAKI [4]: Assuming that μ is a Radon measure over a locally compact Hausdorff space X, and Y is a topological space, we say that f is a μ *measurable function in the Bourbakian sense* if and only if f is a function whose image is contained in Y, whose domain is contained in $X, \mu(X\sim \operatorname{dmn} f)=0$, and for which the following condition is satisfied:

For every $\varepsilon>0$ every μ measurable set A with $\mu(A)<\infty$ contains a compact set C so that $\mu(A\sim C)<\varepsilon$, $C\subset \operatorname{dmn} f$ and $f|C$ is continuous.

It is not hard to prove, that if f is a μ measurable function in the Bourbakian sense, then f is μ measurable function. By Lusin's theorem, if Y is a separable metric space, and μ is a Radon measure over a locally compact Hausdorff space X, then every μ measurable function is μ measurable in the Bourbakian sense [see FEDERER [9], 2.3.5 and 2.3.6].

We will say, that f is a μ *measurable function over A* [*μ measurable function over A in the Bourbakian sense*] if and only if μ is a measure over $X, A\subset X$ and f is a ν measurable function [ν measurable function in the Bourbakian sense] where $\nu(B)=\mu(B\cap A)$ for $B\subset X$.

§ 2. The main results

2.1 Definition. Let μ and ν be measures on X and Y respectively, and let T be a set. For every $t\in T$ let $g_t\subset X\times Y$ be a relation. We shall say, that the family of relations g_t, $t\in T$ is *(μ, ν)-continuous in measure uniformly in $t\in T$*, if for each $\varepsilon>0$ there exists a $\delta>0$ so that $A\subset X$ and $\mu(A)<\delta$ imply $\nu(g_t(A))<\varepsilon$ for every $t\in T$.

In case confusion is excluded we shall say, that g_t, $t\in T$ is uniformly continuous in measure. We remark that in the followings we shall always need uniform continuity of families of relations represented as inverses of functions.

2.2 Remarks. The concept of uniform continuity in measure of a family of relations is related to the concept of absolute continuity of measures. Indeed, with the notations used in 2.1, if for every $t\in T$, g_t^{-1} is a function, and

$$\varkappa(A) = \sup\{\nu(g_t(A)) \colon t\in T\} \quad \text{whenever} \quad A\subset X,$$

than $\varkappa$ is a measure over X and g_t, $t\in T$ is uniformly continuous in measure if and only if the following condition holds:

For every $\varepsilon>0$ there exists a $\delta>0$, that $A\subset X$,

$$\mu(A)<\delta \quad \text{implies} \quad \varkappa(A)<\varepsilon.$$

This condition implies, that $\varkappa(A)=0$ whenever $A\subset X$ and $\mu(A)=0$, but the converse is not true without further conditions (See HALMOS [10]). These account for our use of the notion "uniform continuity in measure".

2.3 Theorem. *Let T, Y and X_i $(i=1, 2, \ldots, n)$ be sets, ν and μ_i be measures on Y and X_i, respectively, and suppose, that $\mu_i(X_i)<\infty$ $(i=1, 2, \ldots, n)$. Further let $D\subset T\times Y$, and $f_0: T\to\mathbf{R}$, $f_i: X_i\to\mathbf{R}$ $(i=1, 2, \ldots, n)$, $g_i: D\to X_i$ $(i=1, 2, \ldots, n)$, $h: D\times\mathbf{R}^n\to\mathbf{R}$ be functions. Assume, that the following conditions hold:*

(1) *For every $(t, y)\in D$*

$$|f_0(t)| \leqq \left|h\big(t, y, f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))\big)\right|.$$

(2) *The functions $|f_i|$ $(i=1, 2, \ldots, n)$ are μ_i measurable.*

(3) *With the notation*

$$g_{i,t}(y) = g_i(t, y) \quad \text{if} \quad (t, y)\in D \quad \text{and} \quad i = 1, 2, \ldots, n$$

the families of relations $g_{i,t}^{-1}$, $t\in T$ are uniformly continuous in measure.

(4) *There exists an $\varepsilon>0$ so that $\nu(y:(t, y)\in D\}\geqq\varepsilon$ for every $t\in T$.*

(5) *For every $k>0$ there is a $K>0$ such that $|r_i|\leqq k$ $(i=1, 2, \ldots, n)$ and $(t, y)\in D$ implies*

$$|h(t, y, r_1, \ldots, r_n)| \leqq K.$$

Then f_0 is bounded on T.

PROOF. By (4) there exists an $\varepsilon>0$ so that $\nu\{y: (t, y)\in D\}\geqq\varepsilon$ for every $t\in T$. By (3) there exists a $\delta>0$ for which $A_i\subset X_i$ and $\mu_i(A_i)<\delta$ implies $\nu\big(g_{i,t}^{-1}(A_i)\big)<\dfrac{\varepsilon}{n}$ for every $t\in T$ and $i=1, 2, \ldots, n$. As $|f_i|$ is μ_i measurable and $\mu_i(X_i)<\infty$,

$$B_{i,k} = \{x\colon x\in X_i \text{ and } |f_i(x)| > k\}$$

is a decreasing sequence of measurable sets in k with an empty intersection, also there exists k, so that

$$\mu_i(X_i \sim C_i) < \delta \quad \text{for} \quad i = 1, 2, \ldots, n$$

with the notation

$$C_i = \{x\colon x\in X_i,\ |f_i(x)| \leqq k\}.$$

Let t be an element of T. We will prove that the set

$$\bigcap_{i=1}^{n} g_{i,t}^{-1}(C_i) \tag{6}$$

is nonvoid. Since the sets $g_{i,t}^{-1}(X_i\sim C_i)$ and $g_{i,t}^{-1}(C_i)$ are disjoint with union $\{y: (t, y)\in D\}$, supposing (6) being empty, we should have

$$\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i \sim C_i) = \{y\colon (t, y)\in D\}$$

and hence

$$\varepsilon \leqq \nu\left(\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i \sim C_i)\right) \leqq \sum_{i=1}^{n} \nu\big(g_{i,t}^{-1}(X_i \sim C_i)\big) < \varepsilon$$

a contradiction, consequently (6) is nonvoid. Let y be a member of (6). Then $g_{i,t}(y)\in C_i$, hence $|f_i(g_i(t, y))|\leqq k$, consequently by (1) and (5) there exists a $K>0$, for which $|f_0(t)|\leqq K$ and K does not depend on t. This proves, that f_0 is bounded on T, and thus the proof is complete.

2.4 Remarks. The theorem 2.3 can be generalized for the case in which the image of f_i is in a metric space Z_i metrized by ϱ_i for $i=0, 1, \dots, n$. The conditions 2.3(3) and 2.3(4) are unchanged, the conditions 2.3(1), 2.3(2) and 2.3(5) are to be replaced by the followings:

(1) There exists a $z_0\in Z_0$, that for every $(t, y)\in D$ the distance between z_0 and $f_0(t)$ is not larger than the distance between z_0 and

$$h\big(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))\big)$$

(2) For $i=1, 2, \dots, n$ there exists a $z_i\in Z_i$ for which the real valued functions

$$x \to \varrho_i(z_i, f_i(x))$$

are μ_i measurable.

(5) For every $k>0$ there exists $z_i\in Z_i$ $(i=0, 1, \dots, n)$, and $K>0$, so that the distance between z_0 and $h(t, y, z_1', \dots, z_n')$ is not larger than K whenever $(t, y)\in D$, $z_i'\in Z_i$ $(i=1, 2, \dots, n)$ and the distance between z_i and z_i' not larger than k.

This general version of the theorem can be proved easily by the way used in 2.3, or using 2.3.

The theorem can often be used in cases if one of $\mu_i(X_i)<\infty$ $(i=1, 2, \dots, n)$ conditions is not satisfied and $|f_i|$ is μ_i measurable only on a subset A_i of X_i with finite μ_i measure, μ_i being replaced by the measure

$$\nu_i(B) = \mu_i(B\cap A_i) \quad \text{for} \quad B\subset X_i$$

and g_i by a restriction of g_i.

We remark that similar method can be used to prove the upper or lower boundedness of f_0.

2.5 Theorem. *Let Z_0 be a metric space, let Z_i $(i=1, 2, \dots, n)$ be separable metric spaces, T a metric space, X_i $(i=1, 2, \dots, n)$ locally compact metric space and Y be a set with the discrete metric. Let ν be a measure on Y and μ_i be a Radon measure on X_i $(i=1, 2, \dots, n)$. Suppose, that $\mu_i(X_i)<\infty$ $(i=1, 2, \dots, n)$ and $D\subset T\times Y$. Let $f_0\colon T\to Z_0$, $f_i\colon X_i\to Z_i$ $(i=1, 2, \dots, n)$, $g_i\colon D\to X_i$ $(i=1, 2, \dots, n)$, $h\colon D\times Z_1\times Z_2\times\dots\times Z_n\to Z_0$ be functions. Let t_0 be a fixed element of T, and suppose, that the following conditions hold:*

(1) *For every $(t, y)\in D$*

$$f_0(t) = h\big(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))\big).$$

(2) *The functions f_i $(i=1, 2, \dots, n)$ are μ_i measurable.*

(3) *With the notation*

$$g_{i,t}(y) = g_i(t, y) \quad \text{if} \quad (t, y)\in D \quad \text{and} \quad i = 1, 2, \dots, n,$$

the families of relations $g_{i,t}^{-1}$, $t\in T$ are uniformly continuous in measure.

(4) *There exists an* $\varepsilon>0$ *so that*

$$\nu(\{y: (t, y)\in D\}\cap\{y: (t_0, y)\in D\}) \geqq \varepsilon$$

for every $t\in T$.

(5) *For every compact subset* C *of* $Z_1\times Z_2\times\ldots\times Z_n$, h *is uniformly continuous on* $D\times C$.

(6) g_i *is uniformly continuous on* D *for* $i=1, 2, \ldots, n$.

Then f_0 *is continuous at* t_0.

PROOF. By (4) there exists an $\varepsilon>0$ so that

$$\nu(\{y: (t, y)\in D\}\cap\{y: (t_0, y)\in D\}) \geqq \varepsilon$$

for every $t\in T$. By (3) there exists $\delta>0$ for which $A_i\subset X_i$ and $\mu_i(A_i)<\delta$ implies $\nu\big(g_{i,t}^{-1}(A_i)\big)<\dfrac{\varepsilon}{2n}$ for every $t\in T$ and $i=1, 2, \ldots, n$. By Lusin's theorem f_i is μ_i measurable in the Bourbakian sense, hence there exists C_i compact subset of X_i for which $\mu_i(X_i\sim C_i)<\delta$ and $f_i|C_i$ is continuous. Clearly

$$\nu\big(g_{i,t}^{-1}(X_i\sim C_i)\big)<\frac{\varepsilon}{2n} \quad \text{for} \quad t\in T \quad \text{and} \quad i=1, 2, \ldots, n.$$

Let $t\in T$ be fixed. We shall prove, that the set

$$\Big(\bigcap_{i=1}^{n} g_{i,t}^{-1}(C_i)\Big)\cap\Big(\bigcap_{i=1}^{n} g_{i,t_0}^{-1}(C_i)\Big)$$

is nonvoid. If this set were empty, we should have

$$\varepsilon \leqq \nu(\{y: (t, y)\in D\}\cap\{y: t_0, y)\in D\}) =$$

$$= \nu\bigg(\Big(\bigcup_{i=1}^{n} g_{i,t}^{-1}(X_i\sim C_i)\Big)\cup\Big(\bigcup_{i=1}^{n} g_{i,t_0}^{-1}(X_i\sim C_i)\Big)\bigg) \leqq$$

$$\leqq \sum_{i=1}^{n}\nu\big(g_{i,t}^{-1}(X_i\sim C_i)\big)+\sum_{i=1}^{n}\nu\big(g_{i,t_0}^{-1}(X_i\sim C_i)\big)<\varepsilon$$

and thereby a contradiction. Consequently, for every $t\in T$ there exists $y\in Y$ for which $(t, y)\in D$, $(t_0, y)\in D$, $g_i(t, y)\in C_i$ and $g_i(t_0, y)\in C_i$ for $i=1, 2, \ldots, n$.

Let α be a positive number and

$$C=f_1(C_1)\times f_2(C_2)\times\ldots\times f_n(C_n).$$

Since C is compact, by (5) there exists a $\beta>0$ so that the distance between $h(t_0, y, z_1, \ldots, z_n)$ and $h(t, y, z_1', \ldots, z_n')$ is less than α whenever $z_i, z_i'\in f_i(C_i)$, and the distance between t and t_0, z_1 and z_1', z_2 and $z_2', \ldots, z_n$ and z_n' is less than β. Since C_i is a compact subset of X_i and f_i is continuous on C_i, so f_i is uniformly continuous on C_i and so there exists $\gamma>0$ for which the distance between $f_i(x_i)$ and $f_i(x_i')$ is less than β whenever $x_i, x_i'\in C_i$ and the distance between them is less, than γ. By (6) there exists a neighbourhood W of t_0, that the distance between

t and t_0 is less than β, and the distance between $g_i(t, y)$ and $g_i(t_0, y)$ is less than γ whenever $t \in W$, $(t, y) \in D$ and $(t_0, y) \in D$.

Let t be an element of W and let's choose y so, that the condition $(t, y) \in D$, $(t_0, y) \in D$, $g_i(t, y) \in C_i$, $g_i(t_0, y) \in C_i$ be satisfied for $i = 1, 2, \dots, n$. Then for $i = 1, 2, \dots, n$ the distance between $f_i(g_i(t, y))$ and $f_i(g_i(t_0, y))$ is less, than β, hence the distance between

$$h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y)))$$

and

$$h(t_0, y, f_1(g_1(t_0, y)), \dots, f_n(g_n(t_0, y)))$$

less than α. Accordingly, the distance between $f_0(t)$ and $f_0(t_0)$ is less than α whenever $t \in W$, thus f_0 is continuous at t_0, and this completes the proof.

2.6 Remarks. Theorem 2.5 can be generalized for the case in which Z_i is a uniform space with the uniformity τ_i $(i = 0, 1, \dots, n)$, T is a topological space, and X_i is a locally compact Hausdorff uniform space with the uniformity σ_i. The conditions 2.5(1), 2.5(3), 2.5(4) are unchanged, the conditions 2.5(2), 2.5(5) and 2.5(6) are to be replaced by the followings:

(2) The functions f_i $(i = 1, 2, \dots, n)$ are μ_i measurable in the Bourbakian sense.

(5) If $V_0 \in \tau_0$ and C is a compact subset of $Z_1 \times Z_2 \times \dots \times Z_n$, then there is a neighbourhood V of t_0 and $V_i \in \tau_i$ $(i = 1, 2, \dots, n)$ such that

$$(h(t, y, z_1, \dots, z_n),\ h(t_0, y, z_1', \dots, z_n')) \in V_0$$

whenever $t \in V$, $(z_i, z_i') \in V_i$ $(i = 1, 2, \dots, n)$, $(t, y) \in D$, $(t_0, y) \in D$ $(z_1, \dots, z_n) \in C$ and $(z_1', \dots, z_n') \in C$.

(6) If $U_i \in \sigma_i$ $(i = 1, 2, \dots, n)$, then there exists a neighbourhood S of t_0, such that $t \in S$, $(t, y) \in D$ and $(t_0, y) \in D$ implies

$$(g_i(t, y),\ g_i(t_0, y)) \in U_i \quad \text{for} \quad i = 1, 2, \dots, n.$$

This theorem can often be used in cases when $A_i \subset X_i$, $\mu_i(A_i) < \infty$ and f_i is μ_i measurable over A_i in the Bourbakian sense, replacing g_i by one of its restrictions.

2.7 A special case. In 2.7 and 2.8 we shall treat the important special case in which the functions $g_{i,t}$ are mapping an open subset of $\mathbf{R}^k$ into $\mathbf{R}^k$. These results are not so general as 2.3, 2.4, 2.5 and 2.6, but it can often be applyed more easily. In the proof of the theorem we shall use the following proposition, which can be found in FEDERER [9], 2.2.7.

2.7.1 Proposition. *In case X is a convex subset of a normed vectorspace, f is a function mapping X into a normed vectorspace Y, and M is a positive real number, the conditions*

$$|f(x) - f(z)| \leqq M \cdot |x - z| \quad \textit{for every} \quad x, z \in X$$

and

$$\limsup_{z \to x} \frac{|f(z) - f(x)|}{|z - x|} \leqq M \quad \textit{for every} \quad x \in X$$

are equivalent. [*Here* $|\ |$ *is the norm on* X *or* Y.]

2.7.2 Theorem. *Let T be a locally compact metric space, let Z_0 be a metric space, and let Z_i $(i=1, 2, \ldots, n)$ be separable metric spaces. Suppose, that D is an open subset of $T\times\mathbf{R}^k$ and $X_i\subset\mathbf{R}^k$ for $i=1, 2, \ldots, n$. Let $f_0\colon T\to Z_0$, $f_i\colon X_i\to Z_i$, $g_i\colon D\to X_i$, $h\colon D\times Z_1\times Z_2\times\ldots\times Z_n\to Z_0$ be functions. Suppose, that the following conditions hold:*

(1) *For every $(t, y)\in D$*

$$f_0(t) = h\big(t, y, f_1(g_1(t, y)), \ldots, f_n(g_n(t, y))\big).$$

(2) *f_i are Lebesgue measurable over X_i for $i=1, 2, \ldots, n$.*

(3) *h is continuous on compact sets.*

(4) *For $i=1, 2, \ldots, n$ g_i is continuous, and for every fixed $t\in T$ the mapping $y\to g_i(t, y)$ are differentiable with the derivate $D_2g_i(t, y)$ and with the Jacobian $J_2g_i(t, y)$, moreover, the mapping $(t, y)\to D_2g_i(t, y)$ is continuous on D and for every $t\in T$ there exists a $(t, y)\in D$ so that*

$$J_2g_i(t, y)\neq 0 \quad \textit{for} \quad i = 1, 2, \ldots, n.$$

Then f_0 is continuous on T.

PROOF. We prove, that if $(t_0, y_0)\in D$ and $J_2g_i(t_0, y_0)\neq 0$ for $i=1, 2, \ldots, n$, then f_0 is continuous at t_0. Let L_i be the inverse of the linear operator $D_2g_i(t_0, y_0)$. By (4) the functions $(t, y)\to D_2g_i(t, y)$ are continuous mappings from D into the space of linear transformations of $\mathbf{R}^k$ into itself provided by the operator norm $\|\ \|$. Hence there exists $\delta>0$ and a compact subset T^* of T containing a neighbourhood of t_0, so that $(t, y)\in D$,

$$\|L_i\circ D_2g_i(t, y)-I\|\leqq\frac{1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n,$$

where I is the identical mapping of $\mathbf{R}^k$ onto itself, and

$$\delta\leqq|J_2g_i(t, y)| \quad \text{for} \quad i = 1, 2, \ldots, n,$$

whenever $t\in T^*$ and $|y-y_0|<\delta$. [Here $|\ |$ is the usual norm on $\mathbf{R}^k$.]

Let $Y^*=\left\{y\colon y\in\mathbf{R}^k \text{ and } |y-y_0|\leqq\frac{\delta}{2}\right\}$, $D^*=T^*\times Y^*$, $Z_i^*=Z_i$ $(i=0, 1, \ldots, n)$, $X_i^*=g_i(D^*)$ $(i=1, 2, \ldots, n)$, $\nu^*=\lambda|2^{Y^*}$, $\mu_i^*=\lambda|2^{X_i^*}$, $(i=1, 2, \ldots, n)$, $f_i^*=f_i|X_i^*$ $(i=1, 2, \ldots, n)$, $f_0^*=f_0|T^*$ $g_i^*=g_i|D^*$, and let h^* be the restriction of h onto $D^*\times Z_1\times Z_2\times\ldots\times Z_n$. Clearly μ_i^* is a Radon measure on X_i^* and $\mu_i^*(X_i^*)<\infty$ since $g_i^*(D^*)$ is compact.

It is now possible for us to complete the proof using theorem 2.5 with sets, measures and functions denoted by *.

The conditions 2.5(1), (2), (4), (5) and (6) are obviously satisfied, and so it is sufficient to prove the existence of positive real numbers c_i so that

$$\lambda\big(g_{i,t}^{-1}(A)\big)\leqq c_i\cdot\lambda(A)$$

whenever $t\in T^*$ and $A\subset g_{i,t}(Y^*)$. First we prove, that

$$\{y\colon y\in\mathbf{R}^k,\ |y-y_0|<\delta\}$$

is mapped by $g_{i,t}$ into $\mathbf{R}^k$ one-to-one, whenever $t \in T^*$. Let i and $t \in T^*$ be fixed, and let the function $G: Y^* \to \mathbf{R}^k$ be defined by

$$G(y) = L_i(g_i(t, y)) - y \quad \text{for} \quad |y - y_0| < \delta.$$

Since the derivate of G at y is equal with $L_i \circ D_2 g_i(t, y) - I$, having a norm not greater than $\frac{1}{2}$, using the previous proposition we have

$$|G(y) - G(z)| \leqq \frac{1}{2} |y - z| \quad \text{if} \quad |y - y_0| < \delta \quad \text{and} \quad |z - y_0| < \delta.$$

Hence

$$\left|L_i(g_i(t, y)) - L_i(g_i(t, z))\right| = \left|L_i(g_i(t, y) - g_i(t, z))\right| \geqq \frac{1}{2} \cdot |y - z|$$

and so

$$|g_i(t, z) - g_i(t, y)| \geqq \frac{|y - z|}{2\|L_i\|} \quad \text{if} \quad |y - y_0| < \delta \quad \text{and} \quad |z - y_0| < \delta.$$

Next we apply the formula concerning transformation of integrals [see 3.3.1, or Federer [9] 3.2.1, 3.2.3 and 3.2.5] with the inverse of the restriction of $g_{i,t}$ onto

$$\{y: y \in \mathbf{R}^k, \ |y - y_0| < \delta\}.$$

We get that

$$\lambda(g_{i,t}^{-1}(A)) \leqq \frac{1}{\delta} \cdot \lambda(A) \quad \text{whenever} \quad t \in T^* \quad \text{and} \quad A \subset g_{i,t}(Y^*),$$

and this completes the proof.

2.8 Remarks

2.8.1 We get an other form of 2.7.2, if we omit the condition "T is locally compact" but suppose, that $\lambda(X_i) < \infty$ and X_i is Lebesgue measurable, and replace 2.7.2(3) by

(3) For every compact subset C of $Z_1 \times Z_2 \times \ldots \times Z_n$, h is uniformly continuous on $D \times C$.

The proof is like that of theorem 2.7.2.

2.8.2 Using 2.6 instead 2.5, the theorem 2.7.2 can be generalized for the case in which (Z_i, τ_i) is an uniform space $(i = 0, 1, \ldots, n)$ and T is a locally compact topological space. The conditions 2.7.2(1) and 2.7.2(4) are unchanged, the conditions 2.7.2(2) and (3) are to be replaced by the followings:

(2) f_i is Lebesgue measurable over X_i in the Bourbakian sense.

(3) For every $t_0 \in T$, $V_0 \in \tau_0$ and compact subset C of $Z_1 \times Z_2 \times \ldots \times Z_n$ there exist a neighbourhood V of t_0 and $V_i \in \tau_i$ $(i = 1, 2, \ldots, n)$ so that

$$(h(t, y, z_1, \ldots, z_n), h(t_0, y, z_1', \ldots, z_n')) \in V_0$$

$$\text{whenever } t\in V,\ (z_i, z_i')\in V_i\ (i=1, 2, \dots, n),\ (t, y)\in D, (t_0, y)\in D,$$
$$(z_1, z_2, \dots, z_n)\in C \quad\text{and}\quad (z_1', z_2', \dots, z_n')\in C.$$

2.8.3 We get an other form of 2.8.2, if we omit the condition "T is locally compact", but suppose, that $\lambda(X_i)<\infty$ and X_i is Lebesgue measurable.

§ 3. Families of relations uniformly continuous in measure

3.1 Remarks. The following remarks give evidence that uniform continuity in measure of a family of relations may be reduced to uniform continuity in measure of simpler families of relations or those of functions. The above mentioned is necessary to state, because in points 3.2 and 3.3 conditions valid for uniform continuity in measure of families of functions only are given.

The proofs are simple calculations, thus, they are omitted. Let μ and ν be measures over X and Y, respectively, and let T be a set.

3.1.1 If $g_{1,t}\subset g_{2,t}\subset X\times Y$ for every $t\in T$, and the family of relations $g_{2,t}$, $t\in T$ is uniformly continuous in measure, then the family of relations $g_{1,t}$, $t\in T$ will be uniformly continuous in measure.

3.1.2 If for $i=1, 2, \dots, n$ $g_{i,t}\subset X\times Y$, and the families of relations $g_{i,t}$, $t\in T$ are uniformly continuous in measure then the family of relations $\bigcup_{i=1}^{n} g_{i,t}$, $t\in T$ will be uniformly continuous in measure.

3.1.3 Let $g_t\subset X\times Y$ for every $t\in T$. If for each $\varepsilon>0$ and $t\in T$ there exist relations $g_{1,t}$ and $g_{2,t}$ such that $g_t=g_{1,t}\cup g_{2,t}$, $\nu(g_{1,t}(X))<\varepsilon$ and the family of relations $g_{2,t}$, $t\in T$ is uniformly continuous in measure then the family of relations g_t, $t\in T$ is uniformly continuous in measure.

3.1.4 Let $g_t\subset Y\times X$ be a function for every $t\in T$. The family of relations g_t^{-1}, $t\in T$ is uniformly continuous in measure if and only if the following assertion is true:

For every $\varepsilon>0$ there exists a $\delta>0$ that $B\subset Y$, $\nu(B)\geqq\varepsilon$ and $t\in T$ implies $\mu(g_t(B))\geqq\delta$.

3.2 Families of real functions. We will give sufficient conditions for uniform continuity in measure of families of real functions.

If $g\subset\mathbf{R}\times\mathbf{R}$ a function and x is an interior point of $\operatorname{dmn} g$, we will use the Dini-derivates

$$D^+g(x) = \liminf_{h\downarrow 0}\frac{g(x+h)-g(x)}{h}$$

$$D_-g(x) = \limsup_{h\uparrow 0}\frac{g(x+h)-g(x)}{h}.$$

We need the following statement, which is a simple consequence of Hewitt and Stromberg [12], (18.48).

3.2.1 Let $c \geqq 0$ be a real number and let $g \subset \mathbf{R} \times \mathbf{R}$ be a function with an open domain. Supposing that $D^+ g(x) \leqq c$ and $D_- g(x) \geqq -c$ for every $x \in A \cap \operatorname{dmn} g$, where $A \subset \mathbf{R}$, we have

$$\lambda(g(A)) \leqq c \cdot \lambda(A).$$

The following theorem gives a general condition for the uniform continuity in measure of families of real functions.

3.2.2 Theorem. *Let T be a nonvoid set, and for every $t \in T$ let $g_t \subset \mathbf{R} \times \mathbf{R}$ be a function with an open domain. Supposing that $D^+ g_t(x)$ and $D_- g_t(x)$ are finites for every $t \in T$ and $x \in \operatorname{dmn} g_t$, moreover, that there exists a natural number L such that*

$$\sum_{n=L}^{\infty} b_n < \infty$$

where $b_n = \sup\{\lambda\{x\colon x \in \operatorname{dmn} g_t,\ D^+ g_t(x) > n \text{ or } D_- g_t(x) < -n\}\colon t \in T\}$.

Then the family of relations g_t, $t \in T$ is (λ, λ)-continuous in measure uniformly in $t \in T$.

Proof. Let ε be a positive real number. Since $\sum_{n=L}^{\infty} b_n < \infty$, there exists a natural number $M \geqq L$, for which

$$\sum_{n=M}^{\infty} b_n < \frac{\varepsilon}{4}.$$

Moreover, there exists a natural number $N \geqq M$, such that $N \cdot b_N < \frac{\varepsilon}{4}$, since supposing $b_n \geqq \frac{\varepsilon}{4n}$ for every $n \geqq M$ we should have

$$\sum_{n=M}^{\infty} b_n \geqq \sum_{n=M}^{\infty} \frac{\varepsilon}{4n} = \infty$$

a contradiction. Let $\delta = \frac{\varepsilon}{2N}$. We will prove that $A \subset \mathbf{R}$ and $\lambda(A) < \delta$ implies $\lambda(g_t(A)) < \varepsilon$ for every $t \in T$.

Let t be fixed. If $n = N, N+1, \ldots,$ let

$$A_n = \{x\colon x \in \operatorname{dmn} g_t,\ D^+ g_t(x) > n \text{ or } D_- g_t(x) < -n\}.$$

Clearly, $A_N \supset A_{N+1} \supset \ldots,\ \bigcap_{n=N}^{\infty} A_n = \emptyset$ and $\lambda(A_n) \leqq b_n$ if $n = N, N+1, \ldots.$ Since λ is a Radon measure, there exists a sequence B_n, $n = N, N+1, \ldots,$ of λ measurable sets, for which $B_N \supset B_{N+1} \supset \ldots,\ B_n \supset A_n$ and $\lambda(B_n) = \lambda(A_n)$ if $n = N, N+1, \ldots.$ Replacing B_n by $B_n \sim \left(\bigcap_{i=N}^{\infty} B_i \sim A_n\right)$ we assume $\bigcap_{n=N}^{\infty} B_n = \emptyset$. Let $C_N = \mathbf{R} \sim B_N$ and for $n = N+1, N+2, \ldots$ let $C_n = B_{n-1} \sim B_n$. Clearly the C_n's are disjoint, λ measurable, $\bigcup_{i=N}^{\infty} C_i = \mathbf{R}$, $C_n \cap A_n = \emptyset$ and $B_n = \bigcup_{i=n+1}^{\infty} C_i$ if $n = N, N+1, \ldots.$ Using 3.2.1, we have that $n \geqq N$ and $A \subset C_n$ implies $\lambda(g_t(A)) \leqq n \cdot \lambda(A)$. Let A be a subset of $\mathbf{R}$ with $\lambda(A) < \delta$. Then

$$\lambda(g_t(A)) \leqq \sum_{n=N}^{\infty} \lambda(g_t(A \cap C_n)) \leqq \lambda(g_t(A \cap C_N)) + \sum_{n=N+1}^{\infty} \lambda(g_t(A \cap C_n)) \leqq$$

$$\leqq N \cdot \lambda(A \cap C_N) + \sum_{n=N+1}^{\infty} \lambda(g_t(C_n)) \leqq N \cdot \delta + \sum_{n=N+1}^{\infty} n \cdot \lambda(C_n) \leqq$$

$$\leqq \frac{\varepsilon}{2} + N \cdot \sum_{i=N+1}^{\infty} \lambda(C_i) + \sum_{n=N}^{\infty} \sum_{i=n+1}^{\infty} \lambda(C_i) =$$

$$= \frac{\varepsilon}{2} + N \cdot \lambda\left(\bigcup_{i=N+1}^{\infty} C_i\right) + \sum_{n=N}^{\infty} \lambda\left(\bigcup_{i=n+1}^{\infty} C_i\right) = \frac{\varepsilon}{2} + N \cdot \lambda(B_N) +$$

$$+ \sum_{n=N}^{\infty} \lambda(B_n) \leqq \frac{\varepsilon}{2} + N \cdot b_N + \sum_{n=N}^{\infty} b_n < \frac{3\varepsilon}{4} + \sum_{n=M}^{\infty} b_n < \varepsilon.$$

3.2.3 Let T be a set, and for every $t \in T$, let $g_t \subset \mathbf{R} \times \mathbf{R}$ be a function. If we study the (λ, λ)-continuity in measure of the family of relations g_t^{-1}, $t \in T$, it is usually sufficient to use 3.1.1, 3.1.2, 3.1.3 and the following proposition.

Proposition. *Let T be a set, and for every $t \in T$ let g_t be a one-to-one everywhere differenciable function with the open domain S_t. If there exists a positive real number c such that $|g_t'(x)| > c$ for every $t \in T$ and $x \in S_t$, then the family of relations g_t^{-1}, $t \in T$ is (λ, λ)-continuous in measure uniformly in $t \in T$.*

PROOF. Let us observe that for every $t \in T$, $g_t(S_t)$ is an open subset of $\mathbf{R}$ and g_t^{-1} is a one-to-one everywhere differentiable function with domain $g_t(S_t)$ and with the derivate between $-\frac{1}{c}$ and $\frac{1}{c}$. Now use 3.2.2.

3.3 Families of other functions. In this section we will give some conditions for uniform continuity in measure of families of functions mapping metric space into metric space. Theorem 3.3.1 is a well-known and simple consequence of more general theorems [See FEDERER [9], under points 3.2.1, 3.2.5]. It is obvious, that using the more general results instead of 3.3.1, the theorems 3.3.2 and 3.3.3 based on the later 3.3.1 can be more generalized.

3.3.1 Theorem. *Let S be an open subset of $\mathbf{R}^n$* and $g: S \to \mathbf{R}^n$ *a one-to-one everywhere differentiable function with Jacobian $Jg(x)$ at $x \in S$. Then for every λ measurable subset A of S*

$$\lambda(g(A)) = \int_A |Jg(x)| \, d\lambda x.$$

3.3.2 Theorem. *Let T be a nonvoid set, and for every $t \in T$ let g_t an everywhere differentiable one-to-one function with an open domain $S_t \subset \mathbf{R}^n$ and image contained in $\mathbf{R}^n$. Let $Jg_t(x)$ denote the Jacobian of g_t at $x \in S_t$. Let suppose that there exists* *a natural number N such that with*

$$b_n = \sup\{\lambda\{x\colon x\in S_t,\ |Jg_t(x)|>n\}\colon t\in T\},$$

$$\sum_{n=N}^{\infty} b_n < \infty.$$

Then the family of relations g_t, $t\in T$ *is* (λ, λ)*-continuous in measure uniformly in* $t\in T$.

The proof is like that of theorem 3.2.2.

3.3.3 Proposition. *Let* T *be a set, and for every* $t\in T$ *let* g_t *an everywhere continuously differentiable, one-to-one function with an open domain* $S_t\subset \mathbf{R}^n$ *and image contained in* $\mathbf{R}^n$. *Let* $Jg_t(x)$ *denote the Jacobian of* g_t *at* $x\in S_t$, *and suppose, that there exists a* $c>0$ *real number such that* $|Jg_t(x)|\geqq c$ *for every* $t\in T$ *and* $x\in S_t$. *Then the family of relations* g_t^{-1}, $t\in T$ *is* (λ,λ)*-continuous in measure uniformly in* $t\in T$.

The proof is obvious, and we omit it.

Obviously using the known fact about Hausdorff type measures [see FEDERER [9]] we can give many other examples for families of relations uniformly continuous in measure. Using the terminology and one of the theorems of ROGERS [22], p. 53 we give here a further proposition with a simple and so omitted proof:

3.3.4 Proposition. *Let* T *be a set, let* (X, ϱ) *and* (Y, σ) *be metric spaces, and for every* $t\in T$ *let* $S_t\subset Y$ *and* $g_t\colon S_t\to X$ *be a function. Let* $f\colon[0,\infty]\to[0,\infty]$ *be a continuous strictly increasing function with* $f(0)=0$, *and let* h *be a Hausdorff function [that is* $h\in\mathscr{H}$*]. If*

$$\varrho\big(g_t(x), g_t(y)\big)\geqq f\big(\sigma(x,y)\big) \quad \textit{for every} \quad t\in T,\quad x, y\in S_t$$

then the family of relations g_t^{-1}, $t\in T$ *is* $(\mu_{h\circ f}, \mu_h)$*-continuous in measure uniformly in* $t\in T$.

3.3.5 Corollary. *Let* T *be a set and for every* $t\in T$ *let* $g_t\subset\mathbf{R}^n\times\mathbf{R}^n$ *be a function. If there exists a real number* $c>0$ *such that*

$$|g_t(x)-g_t(y)|\geqq c\cdot|x-y| \quad \textit{for every} \quad t\in T,\quad x,y\in \operatorname{dmn} g_t$$

[where $|\ |$ *is the usual norm on* $\mathbf{R}^n$*], then the family of relations* g_t^{-1}, $t\in T$ *is* (λ,λ)*-continuous in measure uniformly in* $t\in T$.

§ 4. Applications

4.1 On a functional equation of the information theory. In [15] it is proved by PL. KANAPPAN and C. T. NG that every Lebesgue measurable function $f\colon]0,1[\to\mathbf{R}$ for which

$$f(t)=f(1-y)+(1-y)f\left(\frac{t}{1-y}\right)-(1-t)f\left(\frac{1-t-y}{1-t}\right) \tag{1}$$

whenever $(t, y)\in D$,

where

$$D = \{(t, y):\ t, y \in]0, 1[\text{ and } t+y < 1\}$$

is continuous. (See in connection with this LEE [20], ACZÉL and DARÓCZY [2]). Clearly, this is a simple consequence of theorem 2.7.2.

4.2 On the functional equation $f(uv)+f((1-u)(1-v))=f(u(1-v))+f(v(1-u))$

Let $f:]0, 1[\to \mathbf{R}$ be an unknown function and consider the functional equation

$$f(uv)+f((1-u)(1-v)) = f(u(1-v))+f(v(1-u)) \tag{1}$$

whenever $u, v \in]0, 1[$.

In what follows we will prove that

$$f(x) = a\cdot x\cdot(1-x)+b\cdot \ln x + c \quad \text{if} \quad x\in]0, 1[$$

is the general Lebesgue measurable solution of (1). (See also LAJKÓ [19] and ELIEZER [8].)

4.2.1 Theorem. *Let f be a Lebesgue measurable solution of* (1). *Then f is infinitely many times differentiable.*

PROOF. Let $t=uv$ and $y=u(1-v)$. With this substitution (1) changes to the functional equation

$$f(t) = -f\left(\frac{y}{t+y}(1-t-y)\right)+f\left(\frac{t}{t+y}(1-t-y)\right)+f(y) \tag{2}$$

for every $(t, y)\in D$,

where $D=\{(t, y): t, y\in]0, 1[\text{ and } t+y<1\}$. Using theorem 2.7.2, we have that f is continuous on $]0, 1[$. Having this, the theorem can be proved by means of generally used methods. [See ACZÉL [1]].

Let t_0 be any element of $]0, 1[$, and let $0<\alpha<\beta<\sqrt{t_0}-t_0$. Then there exist real numbers a_1, a_2, b_1, b_2, c and d, such that $0<c<t_0<d<1$ and for every $t\in[c, d]$

$$[\alpha, \beta]\subset]0, \sqrt{t}-t[\subset]0, 1-t[,$$

$$g_1([c, d]\times[\alpha, \beta])\subset]a_1, b_1[\subset]0, (1-\sqrt{t})^2[$$

$$g_2([c, d]\times[\alpha, \beta])\subset]a_2, b_2[\subset]0, 1-t[$$

where

$$g_1(t, y) = \frac{y}{t+y}(1-t-y) \quad \text{for} \quad (t, y)\in D$$

and

$$g_2(t, y) = \frac{t}{t+y}(1-t-y) \quad \text{for} \quad (t, y)\in D.$$

Let

$$h_1(t, x) = \frac{1-t-x}{2}-\frac{\sqrt{(x-(1+\sqrt{t})^2)(x-(1-\sqrt{t})^2)}}{2}.$$

if $t\in]c, d[$ and $x\in]0, (1-\sqrt{t})^2[$. If a $t\in]c, d[$ is fixed in $h_1(t, x)$, we get the inverse of the mapping $y\to g_1(t, y)$ of $]0, \sqrt{t}-t[$ onto $]0, (1-\sqrt{t})^2[$.

Similarly, let

$$h_2(t, x) = \frac{t}{t+x}(1-t-x).$$

if $t\in]c, d[$ and $x\in]0, 1-t[$. If $t\in]c, d[$ is fixed in $h_2(t, x)$ we get the inverse of the mapping $y\to g_2(t, y)$ of $]0, 1-t[$ onto $]0, 1-t[$. Since if t is a fixed element of $]c, d[$, the functions

$$f\left(\frac{y}{t+y}(1-t-y)\right), \quad f\left(\frac{t}{t+y}(1-t-y)\right) \quad \text{and} \quad f(y)$$

of y are continuous on $[\alpha, \beta]$, from (2) we get

$$\int_\alpha^\beta f(t)\,dy = -\int_\alpha^\beta f\left(\frac{y}{t+y}(1-t-y)\right)dy + \int_\alpha^\beta f\left(\frac{t}{t+y}(1-t-y)\right)dy + \int_\alpha^\beta f(y)\,dy.$$

Hence, with the substitution $x=\frac{y}{t+y}(1-t-y)$ in the first and with the substitution $x=\frac{t}{t+y}(1-t-y)$ in the second member of the right side we get

$$(3)\qquad \begin{aligned}(\beta-\alpha)f(t) = &-\int_{g_1(t,\alpha)}^{g_1(t,\beta)} f(x)\cdot\frac{\partial}{\partial x}h_1(t, x)\,dx+\\ &+\int_{g_2(t,\alpha)}^{g_2(t,\beta)} f(x)\cdot\frac{\partial}{\partial x}h_2(t, x)\,dx+\int_\alpha^\beta f(y)\,dy.\end{aligned}$$

Applying the theorem on differentiablity by parameter of parametric integrals [see for example DIEUDONNE [7], p. 179] we have, that f is continuously differentiable on $]c, d[$ and so at t_0. Since t_0 was arbitrary, f is continuously differentiable on $]0, 1[$. Let us differentiate equation (3) with respect to t. Using again the theorem on differentiablity by parameter of parametric integrals, we get, that f'' is continuous on $]0, 1[$. Repeating this process, and using, that $h_1(t, x)$, $h_2(t, x)$, $g_i(t, \alpha)$ and $g_i(t, \beta)$ $(i=1, 2)$ are infinitely many times differentiable, the proof is complete.

The next theorem is due to KÁROLY LAJKÓ [unpublished].

4.2.2 Theorem. *f is a solution of* (1) *for which f''' is continuous if and only if*

$$(4)\qquad f(x) = a\cdot x\cdot(1-x)+b\cdot\ln x+c$$

where a, b, c are arbitrary real constans.

PROOF. It is easy to see, that every function type (4) is a solution of (1), and we omit this simple calculation.

Let us differentiate equation (1) with respect to v and multiply by $v-1$. Interchanging the variables and substracting the so obtained equation from the original we get

$$(5)\qquad (v-u)f'(uv) = (u-1)f'\big(v(1-u)\big)+(1-v)f'\big(u(1-v)\big).$$

Repeating this with the equation (5) but multiplying by v instead of $v-1$, and add the result to (5) we obtain

$$(6)\qquad 2(v-u)F(uv) = (1-2v)F\big(u(1-v)\big)+(2u-1)F\big(v(1-u)\big)$$

where

$$(7)\qquad F(x) = f'(x)+xf''(x).$$

With the substitution $v=\frac{1}{2}$ in (6) $F\left(\frac{u}{2}\right)=-F\left(\frac{1-u}{2}\right)$, hence

$$(8)\qquad F'\left(\frac{u}{2}\right) = F'\left(\frac{1-u}{2}\right).$$

Let us differentiate equation (6) with respect to v, with the substitution $v=\frac{1}{2}$ and use (8) we have

$$(9)\qquad 4F(t)+(1-4t)F'(t) = 0, \quad \text{where} \quad t=\frac{u}{2}.$$

Solving this differential equation we have that there exists $a\in\mathbf{R}$ for which

$$F(x) = a(1-4x) \quad \text{if} \quad x\in\left]0, \frac{1}{2}\right[.$$

Hence by (7) there exist a, b, c real numbers for which

$$f(x) = a\cdot x\cdot(1-x)+b\cdot\ln x+c \quad \text{for} \quad x\in\left]0, \frac{1}{2}\right[.$$

But if $u, v\in]0, 1[$ and $uv\geqq\frac{1}{2}$, then $u(1-v)$, $v(1-u)$, $(1-u)(1-v)\in\left]0, \frac{1}{2}\right[$, and so by (1)

$$f(x) = a\cdot x\cdot(1-x)+b\cdot\ln x+c \quad \text{for} \quad x\in]0, 1[.$$

4.3 On the cosine functional equation

In this section we will study the measurable solutions of the cosine functional equation with the help of the general theorem 2.6. In connection with this functional equation see BAKER [3], HILLE and PHILLIPS [13], KUREPA [17] and NAGY [21].

Proposition. *Let G be a locally compact Hausdorff group with the following property:*

(1) *G has a compact subset C with positive left Haar measure such that for any compact subset A of C with positive left Haar measure the left Haar measure of the set $\{x^2\colon x\in A\}$ is positive too.*

If H is a topological ring, $f\colon G\to H$, f is left Haar measurable in the Bourbakian sense and

$$(2)\qquad f(uv)+f(uv^{-1}) = 2f(u)f(v) \quad \textit{for every} \quad u, v\in G,$$

then f is continuous on G.

PROOF. From (2), with $t=uv^{-1}$, $y=v$, we have

$$f(t) = 2f(ty)f(y)-f(ty^2) \quad \text{for every} \quad t, y\in G.$$

Let t_0 denote an arbitrary element of G. We prove, that f is continuous at t_0. We will use 2.6. Let T be a compact set containing a neigbourhoud of t_0, let C be the set from (1), $D=T\times C$, and let $Y=X_1=X_2=X_3=G$ with the right uniform structure of G. Let $Z_i=H$ for $i=0, 1, 2, 3$ with the uniform structure of H as an [additive] topological group. Let $\nu=\lambda$, the left Haar measure on G, let $\mu_1(B)=\lambda(T\cap B)$, $\mu_2(B)=\lambda(C\cap B)$, $\mu_3(B)=\lambda(B\cap\{ty^2: t\in T, y\in C\})$, for $B\subset G$, $f_0= =f|T$ and $f_i=f$ for $i=1, 2, 3$. Let $h(t, y, z_1, z_2, z_3)=2z_1z_2-z_3$ for every $(t, y)\in D$ and $z_i\in Z_i$ $(i=1, 2, 3)$. Next let the functions $g_i: D\to X_i$ be defined with

$$g_1(t, y) = ty \quad \text{for} \quad (t, y)\in D$$

$$g_2(t, y) = y \quad \text{for} \quad (t, y)\in D$$

$$g_3(t, y) = ty^2 \quad \text{for} \quad (t, y)\in D.$$

Clearly, the conditions 2.5(1), 2.5(4), moreover 2.6(2), 2.6(5) and 2.6(6) are satisfied. It is easy to see the uniform continuity in measure of the families of relations $g_{1,t}^{-1}$, $t\in T$ and $g_{2,t}^{-1}$, $t\in T$, thus, we can prove this only for $g_{3,t}^{-1}$, $t\in T$.

Since if $B\subset G$, then

$$\{y^2: y\in C,\ ty^2\in B\} = t^{-1}\cdot(B\cap\{ty^2: t\in T,\ y\in C\})$$

for every $t\in T$, it is sufficient to prove that for every $\varepsilon>0$ there exists a $\delta>0$ such that $B\subset\{y^2: y\in C\}$ and $\lambda(B)<\delta$ implies $\lambda\{y: y^2\in B\}<\varepsilon$. Suppose, that this is not true. Then there exists an $\varepsilon_0>0$, and for every natural number n an open subset U_n of G for which $\lambda(U_n)<\frac{1}{2^n}$ and

$$\lambda\{y: y\in C \text{ and } y^2\in U_n\} \geqq \varepsilon_0.$$

Since the mapping $y\to y^2$ is continuous, the sets $\{y: y^2\in U_n\}$ are open, and so the sets $\{y: y^2\in U_n\}\cap C$ are λ measurable.

Let $B_n=\{y^2: y\in C\}\cap\left(\bigcup_{i=n}^{\infty} U_i\right)$. Then $B_1\supset B_2\supset\dots$, $\lambda(B_n)<\frac{1}{2^{n-1}}$ and the sets $\{y: y^2\in B_n\}$ are λ measurable with finite measures which are not less then ε_0. If

$$B = \left\{y: y^2\in\bigcap_{n=1}^{\infty} B_n\right\}, \quad \text{then} \quad \lambda(B)\geqq\varepsilon_0>0,$$

but

$$\lambda\{y^2: y\in B\} = \lambda\left(\bigcap_{n=1}^{\infty} B_n\right) = 0.$$

If A is a compact subset of B with positive left Haar measure we have a contradiction with (1).

Condition (1) is fulfilled in many important locally compact Hausdorff groups, but not in all of them. For example, if $G=\{-1, 1\}$ with multiplication and discrete topology, and $\mathfrak{n}$ is a cardinal number, then $G^{\mathfrak{n}}$ satisfies (1) if and only if $\mathfrak{n}$ is finite.

It is easy to see, that any groups the with discrete topology, $\mathbf{R}^n$, and the complex unit circle with the usual topology satisfy (1). If G_1 and G_2 satisfy (1) with subsets C_1 and C_2 then $G=G_1\times G_2$ satisfies (1) with $C=C_1\times C_2$. Indeed, if A is a compact subset of C and $\lambda(A)>0$, then by Fubini's theorem there exists a compact subset A_1 of C_1 such that $\lambda(A_1)>0$ and $\lambda\{x_2: (x_1, x_2)\in A\}>0$ for every $x_1\in A_1$. Hence $\lambda\{x_2^2: (x_1, x_2)\in A\}>0$ for every $x_1\in A_1$, thus with the notation

$$B=\{(x_1^2, x_2^2): (x_1, x_2)\in A\}$$

for every element y_1 of the set $B_1=\{x_1^2: x_1\in A_1\}$ with positive measure the set $\{y_2: (y_1, y_2)\in B\}$ has a positive measure, e.g. B has a positive measure.

In the followings we shall prove that every Lie group satisfies condition (1). Let G be an n-dimensional Lie group with the unit element e. It is not hard to prove using some theorem on left Haar measure of Lie groups [see, for example CHEVALLEY [5]] that there exist open subsets U, V and a homeomorphism φ of U onto an open subset of $\mathbf{R}^n$ so that $e\in V\subset U$, $\varphi(e)=0$, the mapping $(x, y)\to\varphi(\varphi^{-1}(x)\cdot(\varphi^{-1}(y))^{-1})$ is an analitic mapping of $\varphi(V)\times\varphi(V)$ into $\varphi(U)$ and that for every compact subset A of V the left Haar measure of A and the n-dimensional Lebesgue measure of $\varphi(A)$ vanish together. Since the Jacobian of the mapping

$$x\to\varphi(\varphi^{-1}(x)\cdot\varphi^{-1}(x))$$

of $\varphi(V)$ into $\varphi(U)$ equal 2^n at 0, we have that there exists an open neighbourhood W of e in G so that $W^2\subset V$ and the Jacobian of the mapping $x\to\varphi(\varphi^{-1}(x)\cdot\varphi^{-1}(x))$ over $\varphi(W)$ not less than 1. Hence by the transformation formula of integrals for any compact subset A of W the Lebesgue measure of $\varphi(\{x^2: x\in A\})$ not less than the Lebesgue measure of $\varphi(A)$. Hence G satisfies (1) with any compact subset C of W wich has a positive left Haar measure.

References

[1] J. ACZÉL, Lectures on Functional Equations and Their Applications (Academic Press, *New York—London*, 1966).
[2] J. ACZÉL and Z. DARÓCZY, On Measures of Information and Their Characterizations (Academic Press, 1975).
[3] J. A. BAKER, Regularity Properties of Functional Equations, *Aequationes Math.*, **6** 243—248, 1971.
[4] Н. Бурбаки (N. BOURBAKI), Интегрирование. Мери, интегрирование мер (Наука, Москва, 1967).
[5] C. CHEVALLEY, Theory of Lie Groups I (Princeton University Press, 1964).
[6] Z. DARÓCZY, On the Measurable Solution of a Functional Equation *Acta Math. Acad. Sci. Hungar.*, **14** 95—121.
[7] J. DIEUDONNÉ, Grundzüge der modernen Analysis (VEB Deutscher Verlag der Wissentschaften, Berlin 1971).
[8] C. J. ELIEZER, P116 S1, *Aequationes Math.*, **10**, 311, 1975.

[9] H. FEDERER, Geometric Measure Theory (Springer Verlag Berlin—Heidelberg—New York 1969).
[10] P. HALMOS, Measure Theory (New York, N. Y.: D. Van Nostrand Co. 1950).
[11] E. HEWITT and K. A. ROSS, Abstract Harmonic Analysis Vol. 1 (Springer Verlag Berlin—Heidelberg 1963).
[12] E. HEWITT and K. STROMBERG, Real and Abstract Analysis (Springer Verlag Berlin—New York 1965).
[13] E. HILLE and R. S. PHILLIPS, Functional Analysis and Semi-Groups (Amer. Math. Soc. Coll. Publ., Vol. 31, 1957).
[14] C. IONESCU-TULCEA, Suboperative Functions and Semigroups of Operator *Ark. Mat.*, **4**, 55—61, 1960.
[15] PL. KANNAPPAN and C. T. NG, Measurable Solution of Functional Equations Related to Information Theory *Proc. Amer. Math. Soc.*, **38**, 303—310, 1973.
[16] J. H. B. KEMPERMAN, A General Functional Equation *Trans. Amer. Math. Soc.*, **86**, 28—56, 1957.
[17] S. KUREPA, A Cosine Functional Equation in Hilbert Space *Canad. J. Math.*, **12**, 45—50, 1960.
[18] S. KUREPA, On the Quadratic Functional *Acad. Serbe Sci. Publ. Ins. Math.*, **13**, 57—72, 1959.
[19] K. LAJKÓ, P116, *Aequationes Math.*, **10**, 311, 1975.
[20] P. M. LEE, On the Axioms of Information Theory *Ann. Math. Statist.*, **35**, 415—418.
[21] B. NAGY, Trigonometric Operator Functions (Hungarian) Diss. for candidature of math. science.
[22] C. A. ROGERS, Hausdorff Measures (*Cambridge*, 1970).

(Received February 2, 1976.)