A note on non existence of interpolatory polynomials for (0, 2)-interpolation

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Let there be given n points $x_1, x_2, \ldots, x_n$ such that

$$-1 \leqq x_n < x_{n-1} < \ldots < x_2 < x_1 \leqq 1 \tag{1}$$

and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ be arbitrary real numbers. In 1955 TURÁN and his associates [10] wanted to find out whether or not there is a polynomial $R_n(x)$ of degree $\leqq 2n-1$ such that

$$\begin{cases} R_n(x_\nu) = \alpha_\nu \\ R_n''(x_\nu) = \beta_\nu \end{cases} \quad (\nu = 1, 2, 3, \ldots, n). \tag{2}$$

They termed this kind of interpolation as (0, 2)-interpolation. By choosing x_ν's as the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$, where $P_{n-1}(x)$ stands for the $(n-1)^{th}$ degree Legendre polynomial, they were able to show that if n is odd then the polynomials $R_n(x)$ do not exist in general. But they are uniquely determined when n is even. In other words if n is even and $a_\nu = \beta_\nu = 0$, $\nu = 1, 2, \ldots, n$ then the only polynomial of degree $\leqq 2n-1$ satisfying (2) is identically zero. They also obtained the explicit representation of these polynomials and proved a uniform convergence theorem [1], [2]. Later this uniform convergence theorem was improved by FREUD [3]. Since then many investigators ([4], [5], [6], [8], [9], [13]) have obtained such results by taking the roots of $T_n(x), H_n(x), xL_n'(x)$ and $(1-x^2)P_n^{(\alpha, -\alpha)}(x)$ as nodes of interpolation where $T_n(x)$, $H_n(x)$, $L_n(x)$ and $P_n^{(\alpha, -\alpha)}(x)$ are Tchebychev polynomial of first kind, Hermite polynomial, Laguerre polynomial and Jacobi polynomial with $(\alpha, \beta) = (\alpha, -\alpha)$, of degree n respectively.

Let

$$-1 = x_{n+2} < x_{n+1} < \ldots < x_2 < x_1 = 1 \tag{3}$$

be the zeros of $w_n(x) = (1-x^2)P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. In [7] we proved that there exists a unique polynomial $R_n(x)$ of degree $\leqq 2n+1$ satisfying the conditions:

$$\begin{aligned} R_n(x_\nu) &= \alpha_\nu; \quad \nu = 1, 2, \ldots, n+2 \\ R_n''(x_\nu) &= \beta_\nu; \quad \nu = 2, 3, \ldots, n+1 \end{aligned} \tag{4}$$

provided n is an even positive integer where $\{\alpha_\nu\}_1^{n+2}$ and $\{\beta_\nu\}_2^{n+1}$ are any preassigned values. But for n odd there does not exist, in general, a unique polynomial of degree $\leqq 2n+1$ satisfying the conditions of (4). Recently Varma [12] has proved the following:

Theorem 1. *Given a positive integer n and real numbers*

$$\alpha_k (k=1, 2, \ldots, n+2), \quad \beta_k, \gamma_k (k=2, 3, \ldots, n+1)$$

there is, in general, no polynomial $P_{3n+1}(x)$ of degree $\leqq 3n+1$ such that

$$P_{3n+1}(x_k) = \alpha_k; \quad k=1, 2, \ldots, n+2, \tag{5}$$

$$P'_{3n+1}(x_k) = \beta_k; \quad k=2, 3, \ldots, n+1 \tag{6}$$

and

$$P'''_{3n+1}(x_k) = \gamma_k; \quad k=2, 3, \ldots, n+1 \tag{7}$$

provided x_k's are the zeros of $(1-x^2)T_n(x)$ and if there exists such a polynomial then there is an infinity of them.

It seems that so far no one has obtained such a result for (0, 2)-interpolation. The main object of this note is to find a set of $n+2$ points satisfying (3) for which there does not exist a unique polynomial $Q_{2n+1}(x)$ of degree $\leqq 2n+1$ satisfying (4) regardless whether n is even or n is odd. More precisely we establish the following:

Theorem 2. *If n is a positive integer, $\{x_\nu\}_1^{n+2}$ are the zeros of*

$$(1-x^2)P_n^{\left(-\frac{1}{3}, -\frac{1}{3}\right)}(x), \quad \text{where} \quad P_n^{\left(-\frac{1}{3}, -\frac{1}{3}\right)}(x)$$

is the Jacobi polynomial of degree n with $(\alpha, \beta)=\left(-\frac{1}{3}, -\frac{1}{3}\right)$, and $\{\alpha_\nu\}_1^{n+2}$ and $\{\beta_\nu\}_2^{n+1}$ are arbitrary real numbers then there is, in general, no polynomial $Q_{2n+1}(x)$ of degree $\leqq 2n+1$ such that

$$Q_{2n+1}(x_\nu) = \alpha_\nu; \quad \nu=1, 2, \ldots, n+2 \tag{8}$$

and

$$Q''_{2n+1}(x_\nu) = \beta_\nu; \quad \nu=2, 3, \ldots, n+1, \tag{9}$$

and if there exists such a polynomial then there is an infinity of them.

Proof. Our aim is here to show that when $\alpha_\nu=0$, $\nu=1, 2, \ldots, n+2$ and $\beta_\nu=0$; $\nu=2, 3, \ldots, n+1$ there exists a polynomial $Q_{2n+1}(x)$ of degree $\leqq 2n+1$ which is not identically zero but satisfies (8) and (9). Thus invoking a well known theorem on the solution of a system of equations we prove Theorem 2.

Due to the conditions of the theorem it follows that

$$Q_{2n+1}(x) = (1-x^2)w_n(x)r_{n-1}(x) \tag{10}$$

where $w_n(x)=P_n^{\left(-\frac{1}{3}, -\frac{1}{3}\right)}(x)$ and $r_{n-1}(x)$ is a polynomial of degree $\leqq n-1$. Since $Q''_{2n+1}(x_\nu)=0$ for $\nu=2, 3, \ldots, n+1$ we have

$$(1-x_\nu^2)r'_{n-1}(x_\nu) - \frac{4}{3}x_\nu r_{n-1}(x_\nu) = 0; \quad \nu=2, 3, \ldots, n+1$$

which implies that

$$(1-x^2)\,r_{n-1}'(x)-\frac{4}{3}\,xr_{n-1}(x) = cw_n(x) \tag{11}$$

where c is a non zero numerical constant. Setting

$$r_{n-1}(x) = \sum_{k=0}^{n-1} a_k w_k(x) \tag{12}$$

and using the recurrence relation ([11], formula (4.5.7), p. 72)

$$(1-x^2)\,w_n'(x) = \left(n-\frac{1}{3}\right) w_{n-1}(x) - nxw_n(x) \tag{13}$$

we have from (11):

$$\sum_{k=1}^{n-1}\left(k-\frac{1}{3}\right) a_k w_{k-1}(x) - \sum_{k=0}^{n-1} a_k \left(k+\frac{4}{3}\right) xw_k(x) = cw_n(x). \tag{14}$$

From (14) and formula (4.5.1) of SZEGŐ [11], p. 71 we conclude that

$$cw_n(x) = \sum_{k=1}^{n-2}\left[\frac{k(3k+2)}{6k}\,a_{k+1} - \frac{k(3k+1)(3k-2)}{(6k-5)(3k-1)}\,a_{k-1}\right] w_k(x) -$$

$$-\frac{(3n-2)(n-1)(3n-5)}{(6n-11)(3n-4)}\,a_{n-2}w_{n-1}(x) - \frac{n(3n+1)(3n-2)}{(6n-5)(3n-1)}\,a_{n-1}w_n(x).$$

Now equating the coefficients of $w_k(x)$ on both sides we get

$$\frac{n(3n+1)(3n-2)}{(6n-5)(3n-1)}\,a_{n-1} = -c, \tag{15}$$

$$\frac{(3n-2)(n-1)(3n-5)}{(6n-11)(3n-4)}\,a_{n-2} = 0 \tag{16}$$

and

$$\frac{k(3k+2)}{6k}\,a_{k+1} - \frac{k(3k+1)(3k-2)}{(6k-5)(3k-1)}\,a_{k-1} = 0; \quad k = 1, 2, \ldots, n-2. \tag{17}$$

If n is even then from the above equations (15), (16) and (17) we have $a_{n-2} = a_{n-4} = \ldots = a_2 = a_0 = 0$ and $a_1, a_3, a_5, \ldots, a_{n-1}$ can be determined and are not zero. Similarly, if n is odd we have from the above equations $a_1 = a_3 = a_5 = \ldots = a_{n-2} = 0$ and $a_0, a_2, a_4, \ldots, a_{n-1}$ are not zero and can be determined. Hence regardless whether n is even or odd, in general, there does not exist a unique polynomial $Q_{2n+1}(x)$ of degree $\leqq 2n+1$ satisfying (8) and (9) and there are infinitely many if they exist.

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(Received October 17, 1975.)