Lattice ordered rings

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It is noted in the monograph of L. FUCHS [1] that the general theory of lattice ordered rings is still not developed. It considers predominatly the so-called F-rings which are subrings and sublattices of a complete direct sum of linear ordered rings. With regard to this Fuchs puts the following problem No 34: "When is a ring isomorphic to a lattice ordered ring?, i.e. under what condition does a given ring allow a lattice order."

The present paper gives a necessary and sufficient condition for an extension of the partial order in a ring to be a latticed order. In the particular case of the trivial order $P=\{0\}$ an answer is obtained to Fuchs' problem.

Definition. We shall say, that the partially ordered ring R is lattice ordered, if its additive group is lattice ordered, i.e. for each pair $g, g'\in R$, $g\neq g'$ there exists one $s\in R$ such that $s-g\geqq 0$ $s-g'\geqq 0$ and for any $t\in R$ with $t-g\geqq 0, t-g'\geqq 0$ one has $t-s\geqq 0$.

If R is a directedly ordered ring, then for each $g, g'\in R$, $g\neq g'$ there exists a nonvain set of the elements s of R having the conditions $s-g\geqq 0, s-g'\geqq 0$. It is denoted by $S(g, g')$. Obviously then $H(s-g, s-g')$ is a conic semiring for every $s\in S(g, g')$.

Let R be such a ring, that for every $g, g'\in R$ there exists $S(g, g')\neq\emptyset$. Let $\bar{R}$ be the sign for the set of all ordered pairs of different elements from R, i.e. $\bar{R}=R^2\setminus\operatorname{diag} R$. Then there is a function $\bar{F}$ defined on $\bar{R}$ with values in $S(g, g')$.

Theorem. *The partial order P in the ring R can be extended to a lattice order if and only if it satisfies the following condition:*

(A) *For every $(g, g')\in\bar{R}$ the set $S(g, g')$ is nonvain and there exists a function $\bar{F}$, so that if $(g_1, g_1'), \ldots, (g_n, g_n')\in\bar{R}$ and $s_1, \ldots, s_n$ are their corresponding elements by $\bar{F}$ and if*

$$t_1-g_1,\ t_1-g_1'\in H(P, s_1-g_1, s_1-g_1') = P_1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$t_n-g_n,\ t_n-g_n'\in H(P_{n-1}, s_n-g_n, s_n-g_n') = P_n,$$

then the semiring

$$H(P, s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t_1-s_1, \ldots, t_n-s_n)$$

is a conic semiring (c.s.)

PROOF. The *necessity* is obvios. Indeed let R be a ring, wich admits an order P that can be extended to a lattice order P^*. Then according to the definition there follows that for every $g, g' \in R$, $g \neq g'$ there exists $s \in R$, such that $s-g \geqq 0$, $s-g' \geqq 0$ and if there exist $t \in R$, such that $t-g \geqq 0, t-g' \geqq 0$, then $t \geqq s$. Hence for every $g, g' \in R$, $g \neq g'$ there exists $S(g, g') \neq \emptyset$ and $H(s-g, s-g')$ is a c.s. It also follows that there exists $\bar{F}$ such, that when $(g, g') \in \bar{R}$, then $\bar{F}(g, g') \subseteqq S(g, g')$. Then obviously the semirings

$$P_1 = H(P, s_1-g_1, s_1-g_1'), \ldots, P_n = H(P_{n-1}, s_n-g_n, s_n-g_n')$$

are conic semirings for every $(g_1, g_1'), \ldots, (g_n, g_n') \in \bar{R}$. Hence if $t_i-g_i, t_i-g_i' \in P_i$, $i=1, 2, \ldots, n$ then

$$H(P, s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t_1-s_1, \ldots, t_n-s_n)$$

is a c.s..

Sufficiency. Let R be a ring with partial order P satisfying condition (A). First we shall show that $P^* = H(P, s-g, s-g', t-s)$ where $(g, g') \in \bar{R}$, $s = \bar{F}(g, g')$; $t-g, t-g' \in P$ also satisfies condition (A). Obviously P^* is a c.s.. Let $(g_1, g_1'), \ldots, (g_n, g_n') \in \bar{R}$ and $s_1, \ldots, s_n$ be their corresponding elements by $\bar{F}$. Let $t_1-g, t_1-g_1' \in H(P^*, s_1-g_1, s_1-g_1') = P_1^*, \ldots, t_n-g_n, t_n-g_n' \in H(P_{n-1}^*, s_n-g_n, s_n-g_n') = P_n^*$, then

$$H(P^*, s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t_1-s_1, \ldots, t_n-s_n) =$$
$$= H(P, s-g, s-g', s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t-s, t_1-s_1, \ldots, t_n-s_n)$$

but this is a c.s.. Hence P^* satisfies condition (A).

Let us use the symbol $\sum$ for the set of all extensions of P, which satisfy the condition (A) by means of the same function $\bar{F}$. We shall show that $\cup P_\alpha = \bar{P} \in \sum$, where $P_\alpha \subseteqq P_\beta$, $\alpha < \beta$.

We admit the opposite that $\bar{P}$ does not satisfy the condition (A). That will mean that there exist such $(g_1, g_1'), \ldots, (g_n, g_n') \in \bar{R}$ and if $s_1, \ldots, s_n$ are their corresponding elements by $\bar{F}$, then for some elements

$$t_1-g_1, t_1-g_1' \in H(\bar{P}, \quad s_1-g_1, s_1-g_1') = \bar{P}_1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$t_n-g_n, t_n-g_n' \in H(\bar{P}_{n-1}, s_n-g_n, s_n-g_n') = \bar{P}_n,$$

the semiring

$$H(\bar{P}, s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t_1-s_1, \ldots, t_n-s_n)$$

is not a conic one. This indicates that there exists some $P_{\alpha_i} \in \{P_\alpha\}$ such that $a, -a \in H(P_{\alpha_i}, s_1-g_1, s_1-g_1', \ldots, s_n-g_n, s_n-g_n', t_1-s_1, \ldots, t_n-s_n)$ but this contradicts the condition that (A) is satisfied for each P_α. Hence $\bar{P}$ satisfies (A).

Then according to Zorn's lemma the set $\sum$ have a maximum element Q which will satisfy the condition (A). Let $(g, g') \in \bar{R}$ and s be the corresponding element by $\bar{F}$ and $t-g, t-g' \in Q$. Then $H(Q, s-g, s-g', t-s) \in \sum$ and there are that

$$Q = H(Q, s-g, s-g', t-s)$$

which indicates that Q is a lattice order.

Corollary. *A ring R admits a lattice order if and only if the following condition is satisfied:*

(A) *For every* $(g, g') \in \bar{R}$ *the set* $S(g, g')$ *is nonvain and there exists a function* $\bar{F}$, *so that if* $(g_1, g_1'), \dots, (g_n, g_n') \in \bar{R}$ *and* $s_1, \dots, s_n$ *are their corresponding elements by* $\bar{F}$ *and if*

$$t_1 - g_1,\ t_1 - g_1' \in H(s_1 - g_1,\ s_1 - g_1') = P_1,$$

$$t_2 - g_2,\ t_2 - g_2' \in H(P_1,\quad s_2 - g_2,\ s_2 - g_2') = P_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$t_n - g_n,\ t_n - g_n' \in H(P_{n-1},\ s_n - g_n,\ s_n - g_n') = P_n,$$

then the semiring

$$H(s_1 - g_1,\ s_1 - g_1',\ \dots,\ s_n - g_n,\ s_n - g_n',\ t_1 - s_1,\ \dots,\ t_n - s_n)$$

is a conic one.

The proof follows immediately from the theorem when P is a trivial order, i.e. $P = \{0\}$.

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Literature

[1] L. Fuchs, Partially ordered algebraic systems, 1963.

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