Some remarks on the endomorphism rings of quasi projective modules

By K. M. RANGASWAMY (Canberra)

Dedicated to the memory of Professor A. Kertész

It is shown that if Q is a quasi-projective generator in R-mod and if $\mathrm{End}_R(Q)$ is left perfect, then Q is finitely generated and projective. It is also shown that any Σ-quasi-projective R-module A is a direct sum of countably generated modules and that if A is further artinian, then $\mathrm{End}_R(A)$ is left artinian. These results generalise the theorems of NĂSTĂCESCU [4], KAPLANSKY [3] and HARADA [2] respectively.

All the rings that we consider here are associative with identity and all the modules are unitary left modules. Let R-mod denote the category of all the left R-modules. An R-module Q is said to be quasi-projective if given an epimorphism $\lambda: Q \to Q'$ and any morphism $\alpha: Q \to Q'$, there exists $\mu: Q \to Q$ satisfying $\lambda\mu = \alpha$. Q is said to be Σ-quasi-projective if the direct sum of any number of copies of Q is quasi-projective. In general, a quasi-projective module need not be Σ-quasi-projective. For example, the quotient field K of a complete discrete valuation ring R is one such as an R-module (see 5.9 of [6]). Also a Σ-quasi-projective module need not be projective, as is clear by considering simple non-projective modules. However, for any positive integer n, the direct sum $Q^{(n)}$ of n copies of Q is quasi-projective, whenever Q is. One useful result is that if $P \xrightarrow{\alpha} Q$ is a projective cover of Q ,where Q is quasi-projective, then $\ker\alpha$ is fully invariant in P, that is, is invariant under every endomorphism of P (see [8]). For the general notation, terminology and results we follow [1].

Making rather heavy use of the results of GABRIEL and POPESCU, C. Năstăcescu proved in [4] that if Q is a quasi-projective generator in R-mod and if $\mathrm{End}_R(Q)$ is semi-primary, then Q is finitely generated and projective. We shall now show that this result remains valid if we replace 'semi-primary' by 'left perfect'. Moreover, our proof is much simpler and is based on the following observation by Professor G. Azumaya which may be of independent interest. The proof (by Professor AZUMAYA) is given in detail, since the result has not appeared anywhere.

Lemma 1. (G. AZUMAYA) *If Q is a faithful quasi-projective R-module with a projective cover, then Q is itself projective.*

PROOF. Let $P \xrightarrow{\alpha} Q$ be a projective cover of Q and let $K = \ker\alpha$. We claim that $K = 0$. Suppose $0 \neq k \in K$. Since P is torsion-less, we can find $f: P \to R$ with $f(k) \neq 0$. Since Q is faithful, there exists $q \in Q$ such that $f(k)q \neq 0$. Define $g: R \to Q$ by $g(r) = rq$ for all $r \in R$. Then, by the projectivity of P, we get $h: P \to P$ such that $\alpha h = gf$. Then $\alpha h(k) \neq 0$ and this is a contradiction since K is fully invariant.

Theorem 2. *Let Q be a quasi-projective generator in R-mod and let $S = \mathrm{End}_R(Q)$ be left perfect. Then Q is finitely generated and projective.*

PROOF. Since Q is a generator, R is a direct summand of $Q^{(n)}$, for some positive integer n. Let $e: Q^{(n)} \to R$ be an idempotent projection. Now $\mathrm{End}_R(Q^{(n)}) \cong S_n$, the ring of all $n \times n$ matrices over S so that it is left perfect and this implies that $R \cong eS_n e$ will also be left perfect. Then, as an R-module, $Q^{(n)}$ has a projective cover and it is further faithful and quasi-projective. By Lemma 1, we conclude that $Q^{(n)}$ (and hence Q) is a projective R-module, so that $Q \cong \bigoplus_{i \in I} \mathrm{Re}_i$, $e_i = e_i^2 \in R$ (see [1]). Since S satisfies the descending chain condition on principal right ideals, I is finite, that is, Q is finitely generated and projective.

Remark 3. In the statement of Theorem 2, if we assume, to start with, that Q is finitely generated (finitely presented), then the same conclusion holds under the weaker condition that S is left semi-perfect ($S/J(S)$ is von Neumann regular and idempotents lift modulo the Jacobson radical $J(S)$).

Corollary 4. ([4]) *If Q is a quasi-projective generator in* R-mod *and if* $\mathrm{End}_R(Q)$ *is semi-primary, then Q is finitely generated projective.*

It is known that a finitely generated quasi-projective module is always Σ-quasi-projective. We now show that any Σ-quasi-projective module is made up of countably generated Σ-quasi-projectives. This generalises the famous theorem of I. KAPLANSKY ([3]) on projective modules.

Theorem 5. Any Σ-quasi-projective R-module A is a direct sum of countably generated (Σ-quasi-projective) R-modules.

PROOF. By Σ-quasi-projectivity, A is projective with respect to $\bigoplus_{|A|} A$ and so the canonical epimorphism $\bigoplus_{a \in A} \mathrm{Ra} \to A$ splits. Thus A is a direct summand of a direct sum of cyclic R-modules. By KAPLANSKY [3], A is then a direct sum of countably generated modules which are Σ-quasi-projective.

A module H is said to be hollow if all its proper submodules are small, that is, $A = B + C$ implies that either $A = B$ or $A = C$. In [5], it was shown that a quasi-projective R-module Q is hollow if and only if $\mathrm{End}_R(Q)$ is a local ring. We can sharpen this remark if Q is Σ-quasi-projective. Recall that a module A is said to be local if it has a (unique) maximal proper submodule which contains every proper submodule of A. It is clear that a local module is always cyclic and hollow.

Lemma 6. *A Σ-quasi-projective R-module Q is hollow if and only if Q is local.*

PROOF. Let Q be hollow. From the proof of Theorem 5, it is clear that Q is a direct summand of $\bigoplus_{i \in I} T_i$, where each T_i is cyclic. Since, by [6], $\mathrm{End}_R(Q)$ is local, Q has the exchange property (see [7]) so that $\bigoplus_{i \in I} T_i = Q \oplus \bigoplus_{i \in I} T_i'$, where $T_i' \subseteq T_i$. If we write, for each $i \in I$, $T_i = T_i' \oplus T_i''$, we then have $Q \cong \bigoplus_{i \in I} T_i''$. By the indecomposability of Q, $Q \cong T_i''$ for some i and we are done, since hollow + + cyclic = local.

HARADA ([2]) proved that if P is a projective artinian R-module, then $\mathrm{End}_R(P)$ is a left artinian ring. Lemma 6 enables us to extend Harada's theorem to Σ-quasi-projectives.

Theorem 7. *If A is a Σ-quasi-projective and artinian R-module, then* (i) *A is a direct sum of finitely many local modules and* (ii) *$S=\mathrm{End}_R(A)$ is left artinian.*

PROOF. By Proposition 3.7 of [6], A is a direct sum of finitely many hollow modules. Then (i) follows from Lemma 6. Since A is finitely generated and Σ-quasi-projective, Lemma 2.6 of HARADA [2] holds for A, namely, every left ideal of $S=\mathrm{End}_R(A)$ is of the form $\mathrm{Hom}_R(A, T)$ for a suitable submodule T of A. Then the artinian nature of A implies that S is left artinian.

A module A is said to have finite spanning dimension (for short, f.s.d.), if every non-empty family of non-small submodules of A has a minimal element (see [6]). In view of Lemma 6 and Theorem 7, we extend, in the next proposition, the assertions 3.5 and 3.8 of [6] to Σ-quasi-projectives with little modifications to the original proofs.

Proposition 8. *Let A be a Σ-quasi-projective R-module.* (i) *A has f.s.d. if and only if A is local or artinian.* (ii) *If A has f.s.d., then $S=\mathrm{End}_R(A)$ has f.s.d. as a left S-module.*

References

[1] F. W. ANDERSON and K. R. FULLER, Rings and Categories of modules, *Graduate Text in Math.* **13**, *Berlin—Heidelberg—New York*, (1974).
[2] M. HARADA, On semi-simple abelian categories, *Osaka J. Math.* **7** (1970), 89—95.
[3] I. KAPLANSKY, Projective modules, *Ann. Math.* **68** (1958), 372—377.
[4] C. NĂSTĂSESCU, L'anneau des endomorphismes d'un module torsion, *J. Algebra* **23** (1972), 476—481.
[5] K. M. RANGASWAMY and N. VANAJA, Quasi-projectives in abelian and module categories, *Pacific J. Math.* **43** (1972), 221—238.
[6] K. M. RANGASWAMY, Modules with finite spanning dimension, *Canad. Math. Bull.* **20** (1977), 255—262.
[7] R. B. WARFIELD, JR., Decompositions of injective modules, *Pacific J. Math.* **31** (1969), 263—276.
[8] L. E. T. WU and J. P. JANS, On quasi-projectives, *Illinois J. Math.* **11** (1967), 439—448.

DEPARTMENT OF MATHEMATICS,
INSTITUTE OF ADVANCED STUDIES,
THE AUSTRALIAN NATIONAL UNIVERSITY,
CANBERRA, A.C.T. 2600, AUSTRALIA.

(*Received December 5, 1975.*)