Nagata's metric for uniformities

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J.-I. Nagata's monograph on dimension theory contains his theorem characterizing the dimension of a metric space by property of its metric function ([1], pp. 138—149). J.-I. NAGATA remarks (p. 138) that it is an open problem to find a simpler proof of the theorem. This paper gives a much shorter proof of Nagata's theorem. The relative shortness of our proof is due to formula (2) which helps to avoid Nagata's notations like (9) on p. 142 of [1]. The paper contains also a corollary about n-dimensional uniformities which generate a given n-dimensional metric topology.[1])

The auther wishes to express her warmest thanks to Professor Á. CSÁSZÁR who carefully read the paper and gave valuable advises.

Theorem (NAGATA [1]). *A metric space R has dimension $\leqq n$ iff there exists a topology preserving metric ϱ in R such that, for every $\varepsilon>0$ and for every point $x\in R$,*

$$\varrho\big(S_{\varepsilon/2}(x), y_i\big) < \varepsilon, \quad y_i\in R, \quad i = 1, \dots, n+2 \tag{n}$$

imply $\varrho(y_i, y_j)<\varepsilon$ for some i, j with $i\neq j$.

PROOF. *Necessity.* Let $\dim R\leqq n$; then every open covering A of R can be refined by an open covering consisting at most of $n+1$ discrete collections (see [3], theorem 1.), so there exists an open covering $\mathscr{B}$ such that $\mathscr{B}^{***}<A$ and each $B\in\mathscr{B}^{***}$ intersects at most $n+1$ elements of A (we denote $\mathscr{B}^*=\{S(B, \mathscr{B})\mid B\in\mathscr{B}\}$ and $\mathscr{B}^{***}=((\mathscr{B}^*)^*)^*$).

In view of this we can construct a sequence

$$\mathscr{U}_1 > \mathscr{U}_2^{***} > \mathscr{U}_2 > \mathscr{U}_3^{***} > \dots \tag{1}$$

of open coverings of R such that

(i) mesh $\mathscr{U}_m\to 0$ as $m\to\infty$ and

(ii) $S^2(x, \mathscr{U}_{m+1}^*)$ intersects at most $n+1$ sets of $\mathscr{U}_m$ for every $x\in R$, $m=1, 2, \dots$.

We shall often make use of the fact (easy to check by induction on p) that, for $\mathscr{U}_{m_i}$ from (1), $X\subset R$ and integers $1\leqq m_2<\dots<m_p$,

$$S^2(\dots S^2(X, \mathscr{U}_{m_2}), \dots, \mathscr{U}_{m_p})\subset S^3(X, \mathscr{U}_{m_2}). \tag{2}$$

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[1]) In the case of a topological space R, $\dim R$ is the covering dimension of R; for a uniform space R, ΔdR denotes the large covering dimension of R (see [2], p. 78).

Now we define a new sequence of coverings of R: for $U \in \mathcal{U}_{m_1}$ and integers $1 \leqq m_1 < m_2 < \dots < m_p$ let

$$S_{m_1}(U) = U,$$
$$S_{m_1 \dots m_p}(U) = S^2(\dots S^2(U, \mathcal{U}_{m_2}), \dots, \mathcal{U}_{m_p}),$$
$$\sigma_{m_1 \dots m_p} = \{S_{m_1 \dots m_p}(U) \mid U \in \mathcal{U}_{m_1}\}.$$

It follows from (2) and (1) that $S_{m_1 \dots m_p}(U) \subset S^3(U, \mathcal{U}_{m_2}) \subset S(U, \mathcal{U}_{m_1})$, so we have

$$\sigma_{m_1} = \mathcal{U}_{m_1} < \sigma_{m_1 \dots m_p} < \mathcal{U}^*_{m_1}. \tag{3}$$

It is obvious that $\sigma_{m_1 \dots m_i} < \sigma_{m_1 \dots m_i \dots m_p}$ and easy to check that

$$\frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} \quad \text{always implies} \tag{4}$$

$\sigma_{l_1 \dots l_q} < \sigma_{m_1 \dots m_p}$. In fact, let

$$m_1 = l_1, \dots, m_{i-1} = l_{i-1}, \quad m_i < l_i \tag{5}$$

for some $1 \leqq i \leqq p, q$ and $C = S_{l_1 \dots l_{i-1}}(U)$ ($C = U$ if $i = 1$) for any $U \in \mathcal{U}_{m_1}$. It follows from (2), (1) and (5) that

$$S_{l_1 \dots l_q}(U) \subset S^3(C, \mathcal{U}_{l_i}) \subset S^2(C, \mathcal{U}_{m_i}) \in \sigma_{m_1 \dots m_i} < \sigma_{m_1 \dots m_p}.$$

Let us now define a function $\varrho(x, y)$ over $R \times R$ by

$$\varrho(x, y) = \inf\left\{\frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} \,\middle|\, y \in S(x, \sigma_{m_1 \dots m_p})\right\}, \tag{6}$$

$$\varrho(x, y) = 1 \quad \text{if} \quad y \notin S(x, \sigma_{m_1 \dots m_p}) \quad \text{for every} \quad \sigma_{m_1 \dots m_p}.$$

Since

$$S(x, \mathcal{U}_{m+1}) \subset S_m(x) = \{y \mid \varrho(x, y) < 2^{-m}\} \subset S(x, \mathcal{U}_m), \tag{7}$$

we conclude that $\varrho(x, y) = 0$ iff $x = y$. From (7) and (i) it follows that $\{S_m(x) \mid m = 1, 2, \dots\}$ is, for $x \in R$, a neighbourhood basis of x. Next we prove the triangle axiom for ϱ. Suppose $1 > \varrho(x, y) = a \geqq b = \varrho(z, y)$ (the case of $\varrho(x, y) = 1$ is trivial). In view of (4) and (6) for a given $\varepsilon > 0$ we can choose $\sigma_{m_1 \dots m_p}$ and $\sigma_{l_1 \dots l_q}$ such that

$$\begin{gathered} 2 \leqq p, q; \quad l_1 < m_p, \quad x \in S(y, \sigma_{m_1 \dots m_p}), \quad z \in S(y, \sigma_{l_1 \dots l_q}), \\ a < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}} < a + \varepsilon, \quad b < \frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < b + \varepsilon, \end{gathered} \tag{8}$$

$$\frac{1}{2^{l_1}} + \dots + \frac{1}{2^{l_q}} < \frac{1}{2^{m_1}} + \dots + \frac{1}{2^{m_p}}. \tag{9}$$

It follows from (8) and (9) that $m_i \leqq l_1 \leqq m_{i+1}$ for some $1 \leqq i < p$. First consider the case of $m_i < l_1 < m_{i+1}$. Let $U \in \mathcal{U}_{m_1}$, $V \in \mathcal{U}_{l_1}$,

$$x, y \in S_{m_1 \dots m_p}(U) = A \quad \text{and} \quad z, y \in S_{l_1 \dots l_q}(V) = B. \tag{10}$$

Denote

$D=S_{m_1\dots m_i}(U)$, then $A\subset S^3(D, \mathscr{U}_{m_i+1})\subset S(D, \mathscr{U}^*_{m_i+1})\subset S(D, U^*_{l_1+1})$. Since $B\subset S^3(V, \mathscr{U}_{l_2})\subset S(V, \mathscr{U}^*_{l_1+1})$, so $y\in V$ or $y\in W\in\mathscr{U}^*_{l_1+1}$ and $W\cap V\neq\emptyset$, therefore $A\cup V\subset S(S^2(D, \mathscr{U}^*_{l_1+1}), \mathscr{U}_{l_1})\subset S^2(D, \mathscr{U}_{l_1})$ and hence $x, z\in S_{m_1\dots m_i l_1\dots l_q}(U)$, i.e.

$$\varrho(x, z) \leqq \frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_i}}+\frac{1}{2^{l_1}}+\dots+\frac{1}{2^{l_q}} < a+b+2\varepsilon.$$

Now let $m_i=l_1$ and $1\leqq m_1, \dots, m_i$ be successing integers. If $m_1=1$ we have $\varrho(x, z)\leqq 1=\frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_i}}+\frac{1}{2^{l_1}}<a+b+2\varepsilon$. In the case of $1<m_1$ we conclude from (8) and (3) that $x\in S(y, \mathscr{U}^*_{m_1})$, $z\in S(y, \mathscr{U}^*_{l_1})$ so $x\in S(z, \mathscr{U}^{**}_{m_1})\subset S(z, \mathscr{U}_{m_1-1})=$ $=S(z, \sigma_{m_1-1})$. Hence

$$\varrho(x, z) \leqq \frac{1}{2^{m_1-1}} = \frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_i}}+\frac{1}{2^{l_1}} < a+b+2\varepsilon.$$

It remains to consider the case of $m_i=l_1$ while $m_s, \dots, m_i$ are successing integers but $m_{s-1}+1<m_s$ for some $2\leqq s\leqq i$. Using the notations of (10) and $F=S_{m_1\dots m_{s-1}}(U)$ we conclude that $A\subset S^3(F, \mathscr{U}_{m_s})\subset S(F, \mathscr{U}_{m_s-1})$ and $B\subset S(V, \mathscr{U}_{l_1})\in\mathscr{U}^*_{l_1}=\mathscr{U}^*_{m_i}<\mathscr{U}_{m_s-1}$, hence $A\cup B\subset S^2(F, \mathscr{U}_{m_s-1})=S_{m_1\dots m_{s-1}, m_s-1}(U)$ and therefore

$$\varrho(x, z) \leqq \frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_{s-1}}}+\frac{1}{2^{m_s-1}} = \frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_i}}+\frac{1}{2^{l_1}} < a+b+2\varepsilon.$$

It is clear now that ϱ is a topology preserving metric in R. Let us prove that ϱ is of the property (n). We can choose x_i $(i=1, 2, \dots, n+2)$ and $1\leqq m_1<\dots<m_p$ such that $x_i\in S(y_i, \sigma_{m_1\dots m_p})$ and $\varrho(x, x_i)<\frac{\varepsilon}{2}$, $\varrho(x_i, y_i)<\varepsilon$, $2\varrho(x, x_i)<\frac{1}{2^{m_1}}+\dots$ $\dots+\frac{1}{2^{m_p}}<\varepsilon$. Then for some $U_i\in\mathscr{U}_{m_1}$ we have $x_i, y_i\in S_{m_1\dots m_p}(U_i)\subset S(U_i, \mathscr{U}^*_{m_1+1})$ and $x_i\in S(x, \sigma_{m_1+1,\dots,m_p+1})\subset S(x, \mathscr{U}^*_{m_1+1})$ so $S^2(x, \mathscr{U}^*_{m_1+1})\cap U_i\neq\emptyset$ for $i=1, 2,$ $\dots, n+2$. In view of (ii), for some $i\neq j$, we have $U_i=U_j=W$ so $y_i, y_j\in S_{m_1\dots m_p}(W)$, that is $\varrho(y_i, y_j)\leqq\frac{1}{2^{m_1}}+\dots+\frac{1}{2^{m_p}}<\varepsilon$.

Sufficiency. Let ϱ be a metric in R of the property (n). For a given $\varepsilon>0$ let M be a maximal subset of R such that

$$(11)\qquad \varrho(x, y) \geqq \varepsilon \quad \text{for every} \quad x, y\in M, \quad x\neq y.$$

Now put $\mathscr{A}_\varepsilon=\{\cup\{S_{\varepsilon/2}(x)|x\in S_\varepsilon(y)\}|y\in M\}$. If for some $z\in R$, $\operatorname{ord}\mathscr{A}_\varepsilon(z)>n+1$, then there exist distinct $y_i\in M$ $(i=1, \dots, n+2)$ such that $\varrho(S_{\varepsilon/2}(z), y_i)<\varepsilon$ and hence by (n), $\varrho(y_i, y_j)<\varepsilon$ for some $i\neq j$ which contradicts (11). Therefore $\operatorname{ord}\mathscr{A}_\varepsilon\leqq n+1$ for every $\varepsilon>0$, and so $\Delta d(R, \varrho)\leqq n$. Since $\dim R$ is the minimum of $\Delta d(R, \varrho)$ for all topology preserving metrics ϱ (see [2], theorem 15. on p. 153), we conclude that $\dim R\leqq n$.

Corollary. *Every topology preserving metrizable uniformity on a topological space R of* $\dim R\leqq n$ *can be refined by a topology preserving metric uniformity μ such that* $\Delta d(R, \mu)\leqq n$.

PROOF. The metric ϱ of property (n) gives a uniformity on R with $\Delta dR \leqq n$ and finer than the uniformity induced by the original metric d in R: in view of (i) for every $\varepsilon > 0$ there exists $\mathcal{U}_m$ in (1) such that mesh $\mathcal{U}_m < \varepsilon$ and therefore $\varrho(x, y) < 2^{-m}$, $x, y \in R$ implies $x \in S(y, \mathcal{U}_m)$, i.e. $d(x, y) < \varepsilon$, which completes the proof of the corollary.

Let us note that our corollary gives more information that the theorem 15. ([2], p. 153) about n-dimensional compatible metrizable uniformities on a metric space R of $\dim R \leqq n$. According to the theorem 15. all compatible metric uniformities on R are refined by the same, at most n-dimensional uniformity on R, namely by the fine uniformity of R, which however is not metrizable in general.

References

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(Received December 20, 1975.)