On differentiable iteration groups

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In the present paper we shall give some properties of differentiable and convex iteration groups.

Let the function f fulfil the following hypothesis:

(A) f is defined, continuous and strictly increasing in the interval

$$J = [0, a] \quad \text{and} \quad 0 < f(x) < x \quad \text{for} \quad x \in (0, a].$$

Definition 1. (cf. [4], [8], [10]). A one-parameter family of functions $\{f^t, t \in \ni (-\infty, \infty)\}$ is called to be an iteration group of the function f with respect to zero, provided the following conditions are fulfilled:

1° for every $t \in (-\infty, \infty)$ the function f^t is defined in an interval $[0, \delta_t]$ where $\delta_t > 0$ and $\delta_t = a$, for $t \geqq 0$,

2° for every pair $s, t \in (-\infty, \infty)$

$$f^s(f^t(x)) = f^{t+s}(x) \tag{1}$$

holds for every x for which both sides are meaningful,

3° $$f^1(x) = f(x) \quad \text{for} \quad x \in [0, a].$$

Remark 1. (cf. [10], [6]). If $\{f^t, t \in (-\infty, \infty)\}$ is an iteration group of the function f with respect to zero and f fulfil assumption (A) and if for an $x \in (0, a]$ the function $f^t(x)$ is continuous in $[0, \infty)$, then the function $h(t) \stackrel{\text{df}}{=} f^t(a)$, for $t \in [0, \infty)$ is continuous, strictly decreasing in $[0, \infty)$, $h[[0, \infty)] = (0, a]$ and

$$f^t(x) = \begin{cases} h(t + h^{-1}(x)), & x \in (0, a], \quad t \geqq -h^{-1}(x) \\ 0, & x = 0. \end{cases} \tag{2}$$

The above iteration group as a function of two variables is defined in the set

$$D = [0, a] \times [0, \infty) \cup \{(x, t)\colon\ t < 0,\ 0 \leqq x \leqq h(-t)\} \tag{3}$$

and it is continuous in D.

§ 1. We shall deal with the differentiable iteration groups. Let us start with two definitions:

Definition 2. An iteration group of the function f with respect to zero $\{f^t, t \in (-\infty, \infty)\}$ is called to be of the class C_0^1 if the function $F(x, t) = f^t(x)$ has continuous partial derivatives F'_x, F'_t except on the line $x = 0$.

Definition 3. An iteration group $\{f^t, t\in(-\infty, \infty)\}$ of the function f with respect to zero is called to be of the class C^1_* if it is of class C^1_0 and $f^t(x)$ has the partial derivatives on the line $x=0$ being continuous with respect to each variable.

In all theorems the following general hypothesis is assumed: (H) $\{f^t, t\in\in(-\infty, \infty)\}$ is an iteration group of the function f with respect to zero and f satisfies hypothesis (A).

Throught this note h denotes the function defined by the formula

$$h(t) \stackrel{\text{df}}{=} f(a), \quad t \geqq 0. \tag{4}$$

Theorem 1. *The following equivalences hold:*

(i) *For every $x\in(0, a]$ there exists the limit $\lim_{t\to 0} f^t(x)=x$ iff h is continuous in* $[0, \infty]$

(ii) *$f^t(x)$ is differentiable with respect to t at zero for every $x\in(0, a]$ i.e. there exists the limit*

$$\lim_{t\to 0}\frac{f^t(x)-x}{t} = \left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0} \stackrel{\text{df}}{=} g(x), \quad x\in[0, a] \tag{5}$$

iff h is differentiable in $[0, \infty)$. Then $g(x)\leqq 0$ for $x\in(0, a]$ and $g(0)=0$.

Proof. (i) Let h be continuous. Then, on account of Remark 1, there exists an $s\geqq 0$ such that $x=f^s(a)$. From the definition of h we have $h(t+s)=f^{t+s}(a)==f^t(f^s(a))=f^t(x)$ where $|t|<\delta$ for an $\delta>0$. Hence $\lim_{t\to 0} f^t(x)=h(s)=x$. If $x=0$, then $f^t(0)\equiv 0$.

Conversely, if $\lim_{t\to 0} f^t(x)=x$, for $x\in(0, a]$, then $\lim_{t\to 0} h(t+s)=\lim_{t\to 0} f^t(f^s(a))==f^s(a)=h(s)$, for $s\geqq 0$.

(ii) If there exists the derivative (5) in J, then $\lim_{t\to 0} f^t(x)=x$ in J, so h is continuous. From Remark 1 it follows that $f^t(x)=h(t+h^{-1}(x))$. Hence we have

$$\lim_{t\to 0}\frac{f^t(x)-x}{t} = \lim_{t\to 0}\frac{h(t+h^{-1}(x))-h(h^{-1}(x))}{t}. \tag{6}$$

The existence of one of these limits implies the existence of the other one. Hence h is differentiable in $[0, \infty)$, since $h^{-1}[(0, a]]=[0, \infty)$.

Conversely, if h is differentiable, then $f^t(x)=h(t+h^{-1}(x))$ whence, by relation (6) f^t is differentiable at 0 with respect to t then, by Remark 1, we have $f^t(x)<x$ for $x\in(0, a)$, so $g(x)\leqq 0$ in J.

Corollary 1. *If there exists the derivative $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ in J, then*

$$g(h(t)) = h'(t), \quad t\in[0, \infty). \tag{7}$$

Proof. From Theorem 1, h is differentiable. Moreover we have $g(h(t))==\lim_{s\to 0}\frac{f^s(h(t))-h(t)}{s}=\lim_{s\to 0}\frac{f^{s+t}(a)-f^t(a)}{s}=\lim_{s\to 0}\frac{h(t+s)-h(t)}{s}=h'(t)$.

Theorem 2. *The following equivalences hold:*

(i) *There exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *being continuous in* J *iff* $h\in C^1[[0,\infty)]$ *and* $\lim_{t\to\infty} h'(t)=0$.

(ii) *There exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *being continuous in* J *and differentiable at zero iff* $h\in C^1[[0,\infty)]$ *and there exists the finite limit* $\lim_{t\to\infty}\frac{h'(t)}{h(t)}=\gamma$ *(then* $\gamma=g'(0)$*).*

Proof. Equivalence (i) follows immediately from Corollary 1 and Theorem 1 (ii). Equivalence (ii) follows from (i) and from the relations

$$g'(0)=\lim_{x\to 0}\frac{g(x)}{x}=\lim_{t\to\infty}\frac{g\big(h(t)\big)}{h(t)}=\lim_{t\to\infty}\frac{h'(t)}{h(t)},$$

since the existence of one of these limits implies the existence of the other ones.

Theorem 3. *Suppose that there exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *in* J. *Then for every* $x\in[0,a)$ $f^t(x)$ *is differentiable with respect to* t *in* $\big[-h^{-1}(x),\infty\big)$ *and*

$$\frac{\partial f^t(x)}{\partial t}\overset{\mathrm{df}}{=}\left.\frac{\partial f^s(x)}{\partial s}\right|_{s=t}=g\big(f^t(x)\big). \tag{8}$$

If $f^t(x)$ *is differentiable at* x_0 *for a* t, *then*

$$g(f^t(x_0))=\left.\frac{\partial f^t(x)}{\partial x}\right|_{x=x_0}g(x_0). \tag{9}$$

In particular, if f *is differentiable in* J, *then*

$$g\big(f(x)\big)=f'(x)g(x),\quad \textit{for}\quad x\in J. \tag{10}$$

Proof. (See also [2], [3].) By the definition of g we have

$$g\big(f^t(x)\big)=\lim_{s\to 0}\frac{f^s\big(f^t(x)\big)-f^t(x)}{s}=\lim_{s\to 0}\frac{f^{s+t}(x)-f^t(x)}{s}=\left.\frac{\partial f^s(x)}{\partial s}\right|_{s=t}.$$

If $f^t(x)$ is differentiable with respect to x at x_0 for a t, then in view of (5)

$$g\big(f^t(x_0)\big)=\lim_{s\to 0}\frac{f^s\big(f^t(x_0)\big)-f^t(x_0)}{s}=$$

$$=\lim_{s\to 0}\frac{f^t\big(f^s(x_0)\big)-f^t(x_0)}{f^s(x_0)-x_0}\,\frac{f^s(x_0)-x_0}{s}=\left.\frac{f^t(x)}{x}\right|_{x=x_0}g(x_0),$$

since $\lim_{s\to 0} f^s(x_0)=x_0$.

Corollary 2. *Suppose that there exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *in* J. *Then:*

(i) $g(x_0)\neq 0$ *for an* $x_0\in(0, a]$ *iff for any* $s\geqq -h^{-1}(x_0)$ *there exists the derivative* $\left.\frac{\partial f^s(x)}{\partial x}\right|_{x=x_0}$,

(ii) g *is continuous in* J *and* $g(x)<0$ *for* $x\in(0, a]$ *iff* $\{f^t, t\in(-\infty, \infty)\}$ *is of class* C_0^1.

Proof. (i). If $g(x_0)\neq 0$ and $x_0\neq 0$, then in view of (7) $h'(h^{-1}(x_0))\neq 0$, so the formula $f^t(x)=h(t+h^{-1}(x))$ implies that for $t\geqq -h^{-1}(x_0)$, $f^t(x)$ is differentiable with respect to x at x_0.

Conversely, let $f^t(x)$ be differentiable at x_0 for $t\geqq -h^{-1}(x_0)$ and $g(x_0)=0$. Then, on account of (9), $g(f^t(x_0))=0$ for $t\geqq -h^{-1}(x_0)$. Since $f^t(x_0)=$ $=h(t+h^{-1}(x_0))$ and $h[[0, \infty)]=(0, a]$, we get $g(x)\equiv 0$ in J, but this is impossible.

Equivalence (ii) follows immediately from (i) and Theorem 3.

Remark 2. If there exists the derivative $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ in J and $g(x_0)\neq 0$, then there exists the derivative $f'(x_0)$. However, the converse theorem is not true. There exists an iteration group of the function $f(x)=sx$ for $x\in[0, 1]$, where $0<s<1$ such that there exists the derivative $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ in J and $g(x_0)=0$ for an $x_0>0$.

In fact, there exists a function h strictly decreasing, positive and differentiable in $[0, \infty)$ satisfying the equation

$$h(t+1) = sh(t), \quad \text{for} \quad t\in[0, \infty)$$

and such that $h(0)=1$ and $h'(0)=0$ (see [4]).

The function h defines an iteration group of f by formula (2). This group has the property that there exists $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ and $g(h(n))=0$ for all integers $n\geqq 0$.

Theorem 4. *Suppose that there exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *being continuous in* J, $g(x)<0$ *for* $x\in(0, a]$ *and* g *is differentiable at zero* [1]). *Then* $f^{s'}(0)=$ $=\left.\frac{\partial f^s(x)}{\partial x}\right|_{x=0}=\exp s\ g'(0)=[f'(0)]^s$. *If moreover* $-\infty<g'(0)\neq 0$, *then the iteration group* $\{f^t, t\in(-\infty, \infty)\}$ *is of class* C_*^1.

Proof. We may assume that $s>0$, since $f^{-s}=(f^s)^{-1}$ in a neighbourhood of zero, and $f^0(x)=x$ in J. From Theorem 2 (i) and Corollary 1 it follows that

[1]) We admit $g'(0)=-\infty$ $(\exp(-\infty) \overset{\text{df}}{=} 0)$.

$\ln h\in C^1[[0,\infty)]$, where $h(t)=f^t(a)$. Then in view of Lagrange theorem, for any $t>0$ there exists an $\Theta_t\in(0,s)$ such that

$$\frac{\ln h(t+s)-\ln h(t)}{s}=\frac{h'(t+\Theta_t)}{h(t+\Theta_t)}.$$

Furthermore, according to Theorem 2 (ii) we infer that there exists the limit

$$\lim_{t\to\infty}\frac{h'(t)}{h(t)}=g'(0), \tag{11}$$

so

$$\lim_{t\to\infty}\frac{h'(t+\Theta_t)}{h(t+\Theta_t)}=g'(0).$$

Consequently

$$\lim_{t\to\infty}\ln\frac{h(t+s)}{h(t)}=\lim_{t\to\infty}\big(\ln h(t+s)-\ln h(t)\big)=sg'(0), \tag{12}$$

whence $f^{s'}(0)=\exp sg'(0)$. For $s=1$ we have $f'(0)=\exp g'(0)$, then $f^{s'}(0)=$ $=f'(0)^s$.

Let $-\infty<g'(0)\neq 0$. By Corollary 2 (ii), $\{f^t: t\in(-\infty,\infty)\}$ is of class C_0^1. Moreover, we have shown above that $f^t(x)$ is differentiable with respect to t on the line $x=0$. By (11) and (12), we have the relation

$$\lim_{x\to 0}\frac{\partial f^s(x)}{\partial x}\overset{\text{df}}{=}\lim_{x\to 0}f^{s'}(x)=\lim_{x\to 0}\frac{h'\big(s+h^{-1}(x)\big)}{h'\big(h^{-1}(x)\big)}=\lim_{t\to\infty}\frac{h'(s+t)}{h'(t)}=$$

$$=\lim_{t\to\infty}\frac{h'(t+s)}{h(t+s)}\frac{h(t+s)}{h(t)}\frac{h(t)}{h'(t)}=\lim_{t\to\infty}\frac{h'(t+s)}{h(t+s)}\lim_{t\to\infty}\frac{h(t+s)}{h(t)}\lim_{t\to\infty}\frac{h(t)}{h'(t)}=$$

$$=g'(0)f^{s'}(0)\frac{1}{g'(0)}=f^{s'}(0)=f'(0)^s=\left.\frac{\partial f^s(x)}{\partial x}\right|_{x=0}.$$

The above limits exist, since $g'(0)\neq 0$.

Moreover, $\lim_{x\to 0}\frac{\partial f^s(x)}{\partial s}=\lim_{x\to 0}g\big(f^s(x)\big)=0$, and $\left.\frac{\partial f^s(x)}{\partial s}\right|_{x=0}\equiv 0$. Hence $\{f^t, t\in$ $\in(-\infty,\infty)\}$ is of class C_*^1.

Theorem 5. (Cf. [1], [2].) *The function h defined by* (4) *satisfies the equation*

$$f\big(h(t)\big)=h(t+1),\quad \textit{for}\quad t\in[0,\infty).$$

If h is continuous, then there exists the inverse function $\alpha(x)=h^{-1}(x)$ *defined in* $(0,a]$ *and satisfies the Abel equation*

$$\alpha\big(f(x)\big)=\alpha(x)+1\quad \textit{for}\quad x\in(0,a].$$

If there exists the derivative $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)<0$ *in* $(0,a]$, *then* α *is differentiable,* $\alpha'(x)\neq 0$ *in* $(0,a]$ *and*

$$g(x)=1/\alpha'(x),\quad \textit{for}\quad x\in(0,a].$$

Now, our assertion follows directly from Remark 1 and Corollary 1.

Theorem 6. *Let g be defined and continuous in $[0, a]$, $g(x)<0$, for $x\in(0, a]$ and let the integral*

$$\int_0^a \frac{du}{g(u)} \tag{13}$$

diverge. Then there exists exactly one iteration group $\{f^t, t\in(-\infty, \infty)\}$ of class C_0^1 of a function f satisfying hypothesis (A) *such that*

$$\left.\frac{\partial f^s(x)}{\partial s}\right|_{s=0} = g(x) \quad \text{for} \quad x\in[0, a]. \tag{14}$$

Conversely, if $\{f^t, t\in(-\infty, \infty)\}$ is an iteration group of class C_0^1 of a function f satisfying (A), *then integral* (13) *diverges, where g is given by* (14).

Proof. It is easy to verify, that the function $f^t(x)$ defined by formula (2) where h is a solution of a differential equation (7) such that $h(0)=a$, is an iteration group satisfying the conditions of our theorem. The uniqueness follows from Remark 1 and Corollary 1, since equation (7) has exactly one solution in $[0, \infty)$ such that $h(0)=a$.

Conversely, if $\{f^t, t\in(-\infty, \infty)\}$ is an iteration group of class C_0^1 of a function f satisfying hypothesis (A) then by Corollaries 2 (ii) and 1, $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)<0$, for $x\in(0, a]$ and

$$h^{-1}(x) = -\int_x^a \frac{du}{g(u)} \quad \text{for} \quad x\in(0, a], \tag{15}$$

where $h(t)=f^t(a)$. Hence integral (13) diverges in virtue of the fact that $\lim_{x\to 0} h^{-1}(x)=\infty$.

Remark 3. (Cf. [2], [3].) If $\{f^t, t\in(-\infty, \infty)\}$ is of class C_0^1, then

$$\frac{\partial f^t(x)}{\partial t} = \frac{\partial f^t(x)}{\partial x}\, g(x)$$

in Int D, where D is given by (3).

This equation follows directly from Theorem 3.

We have also the converse

Theorem 7. *If g is continuous in J, $g(x)<0$, for $x\in(0, a]$, $g(0)=0$ and integral* (13) *diverges, then there exists exactly one function F continuous in the set D (given by* (3) *where h^{-1} is defined by* (15)*) and of class C^1 in* Int D *such that*

$$\frac{\partial F(x, t)}{\partial t} = \frac{\partial F(x, t)}{\partial x}\, g(x), \quad (x, t)\in \operatorname{Int} D \tag{16}$$

and $F(x, 0)=x$ for $x\in[0, a]$. This solution F is an iteration group of class C_0^1 such that $\left.\frac{\partial F(x, t)}{\partial t}\right|_{t=0}=g(x)$ in J and $F(x, 1)=f(x)$ fulfils hypothesis (A).

Proof. From Theorem 6 and Remark 3 it follows that there exists an iteration group $\{f^t, t\in(-\infty, \infty)\}$ fulfilling hypothesis (H) of class C_0^1 defined in D and satisfying equation (16).

For every point $(x_0, t_0)\in \operatorname{Int} D$ there exists exactly one solution z of the equation

$$z'(t) = -g(z(t)) \tag{17}$$

defined in the interval $(-\infty, t_0]$, such that $z(t_0)=x_0$. This solution may be uniquely extended to a point $(\bar{t}_0, a)\in Fr\, D$, where $\bar{t}_0\geqq 0$ and $\bar{t}_0\geqq t_0$. From the definition of the set D it follows that $(z(t), t)\in D$ for $t\leqq \bar{t}_0$. If the functions F_1 and F_2 of class C^1 in $\operatorname{Int} D$ satisfy equation (16) and $F_i(x, 0)=x$ for $i=1, 2$ in $[0, a]$, then $F=F_1-F_2$ also satisfies equation (16) and $F(x, 0)=0$ in $[0, a]$. Hence by (17) we have that $F(z(t), t)=0$ for $t\leqq \bar{t}_0$, so $F(x_0, t_0)=0$ and consequently $F\equiv 0$ in D. Thus it follows the uniqueness of solutions of equation (16) in D.

Theorem 8. *Let $f^t(x_0)$ be continuous for an $x_0\in(0, a]$ in $[0, \infty)$. If one of the functions*

$$g_1(x) = \limsup_{t\to 0^+} \frac{f^t(x)-x}{t}, \quad g_2(x) = \limsup_{t\to 0^-} \frac{f^t(x)-x}{t},$$

$$g_3(x) = \liminf_{t\to 0^+} \frac{f^t(x)-x}{t}, \quad g_4(x) = \liminf_{t\to 0^-} \frac{f^t(x)-x}{t}$$

is continuous in $[0, a]$, then there exists the derivative $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ continuous in J.

Proof. Let g_1 be continuous in $[0, a]$. In view of Remark 1, the continuity of $f^t(x_0)$ in $[0, \infty)$ for an $x_0\in(0, a]$ implies that $f^t(x)$ is given by formula (2). Then

$$g_1(x) = \limsup_{t\to 0^+} \frac{f^t(x)-x}{t} = \limsup_{t\to 0^+} \frac{h(t+h^{-1}(x))-h(h^{-1}(x))}{t} = \bar{D}^+ h(h^{-1}(x)),$$

where $\bar{D}^+ h$ denotes the Dini derivative of h. Hence we have the relation

$$g_1(h(t)) = \bar{D}^+ h(t) \quad \text{for} \quad t\geqq 0.$$

Then $\bar{D}^+ h$ is continuous in $[0, \infty)$. Further, on account of Lebesgue theorem (see [5] p. 184), h is absolutely continuous, whence $\bar{D}^+ h=h'$ a.e. in $[0, \infty)$. Moreover, we have

$$h(s) = \int_0^s h'(t)\, dt = \int_0^s \bar{D}^+ h(t)\, dt.$$

Hence, by the continuity of $\bar{D}^+ h$ in $[0, \infty)$, h is of class C^1 in $[0, \infty)$. Then, by Theorem 1 (ii), there exists $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ and $g=g_1$.

§ 2. In the present section we are concerned with the convex iteration groups.

Definition 4. (See [8]) An iteration group $\{f^t, t\in(-\infty, \infty)\}$ with respect to zero is called to be convex (concave) if for any $t\geqq 0$ the functions are convex (concave).

Now we give a short proof of a generalization of A. Smajdor's result (see [9]).

Theorem 9. *Suppose that there exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *in* J. *Then* $\{f^t, t\in(-\infty, \infty)\}$ *is a convex (concave) iteration group iff* g *is convex (concave) in* J.

PROOF. If $\{f^t, t\in(-\infty, \infty)\}$ is a convex iteration group and $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ in J, then

$$g(x)=\lim_{n\to\infty}\left(f^{1/n}(x)-x\right)n, \quad x\in J.$$

Hence, g is convex as a limit of convex functions.

Conversely, let g be convex. Then the right-hand side derivative g'_+ is increasing in J, as well as g is continuous in J and $g(x)<0$ for $x\in(0, a]$. Hence by Theorem 3 and Corollary 2 we have

$$\frac{\partial f^s(x)}{\partial x}=\frac{g\left(f^s(x)\right)}{g(x)}\overset{\mathrm{df}}{=}G_s(x), \quad \text{for} \quad x\in(0, a]. \tag{18}$$

We are going to show that G_s is increasing for any $s\geqq 0$. Let $s>0$. G_s is continuous in $(0, a]$ and right-hand side differentiable in $(0, a)$. Applying formula (18) we get

$$G'_{s_+}(x)=\frac{\frac{\partial_+ g\left(f^s(x)\right)}{\partial x}g(x)-g\left(f^s(x)\right)g'_+(x)}{g(x)^2}=$$

$$=\frac{g'_+\left(f^s(x)\right)\frac{g\left(f^s(x)\right)}{g(x)}g(x)-g\left(f^s(x)\right)g'_+(x)}{g(x)^2}=\frac{g\left(f^s(x)\right)\left[g'_+\left(f^s(x)\right)-g'_+(x)\right]}{g(x)^2}.$$

Since $g(x)<0$, $f^s(x)<x$ for $x\in(0, a]$ and g'_+ is increasing, we get $G'_s+(x)\geqq 0$ in $(0, a)$. Hence G_s is increasing in $(0, a)$ in virtue of Zygmund's lemma (see [5] p. 182). Since $G_s=f^{s'}$, the function f^s is convex in $(0, a)$. Moreover f^s is continuous at 0 and a, whence f^s is convex in J.

The proof for the concave iteration group is the same.

Theorem 10. *Suppose that there exists the derivative* $\left.\frac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *in* J. *If* $f^t(x_0)$ *is convex as a function of* t *for an* $x_0\in(0, a]$ *then* g *is decreasing in* J. *If* g *is decreasing in* J *then for every* $x_0\in(0, a]\, f^t(x_0)$ *is convex.*

PROOF. From Theorem 3 (formula (8)) it follows that $f^t(x_0)$ is convex with respect to t iff $g\left(f^t(x_0)\right)$ is increasing with respect to t. This condition holds iff g is decreasing, since $f^t(x_0)$ is decreasing and $f^t(x_0)$ maps the interval $[-h^{-1}(x_0), \infty)$ onto $(0, a]$, where $h(t)=f^t(a)$.

Theorem 11. *There does not exist an iteration group* $\{f^t, t\in(-\infty, \infty)\}$ *with respect to zero of the function* f *fulfilling assumption* (A) *such that there exists the* *derivative* $\left.\dfrac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ *being differentiable at* $(0, a]$ *and* $f^t(x)$ *as a function of two variables is convex in* $(0, a]\times[0, \infty)$.

PROOF. Suppose that $F(x, t)\stackrel{\text{df}}{=}f^t(x)$ is convex in $[0, a]\times[0, \infty)$ where $\{f^t, t\in(-\infty, \infty)\}$ satisfies assumption of our theorem. F is convex with respect to each variable. On account of Theorem 9 and 10 g is convex and decreasing in $(0, a]$ as well as $g(x)<0$ for $x\in(0, a]$. Moreover g is of class C^1 in $(0, a]$, since g is differentiable and convex in $(0, a]$. Therefore, Theorem 3 implies that there exist the continuous derivatives

$$\frac{\partial F(x, t)}{\partial t}=g(F(x, t)) \quad \text{and} \quad \frac{\partial F(x, t)}{\partial x}=\frac{g(F(x, t))}{g(x)},$$

whence it follows that F is of class C^2 in $(0, a]\times(0, \infty)$.

Applying the criterion of convexity of functions of two variables (see [7] th. 4.5) we get that the matrix

$$\left\|\begin{matrix} \dfrac{\partial^2 F(x, t)}{\partial t^2} & \dfrac{\partial^2 F(x, t)}{\partial t\,\partial x} \\[2ex] \dfrac{\partial^2 F(x, t)}{\partial x\,\partial t} & \dfrac{\partial^2 F(x, t)}{\partial x^2} \end{matrix}\right\|$$

is nonnegatively defined for $x\in(0, a]$ and $t\in(0, \infty)$.

For our function F this is impossible. In fact, using the formulas (8) and (9) we get

$$\begin{vmatrix} \dfrac{\partial^2 F(x, t)}{\partial t^2} & \dfrac{\partial^2 F(x, t)}{\partial t\,\partial x} \\[2ex] \dfrac{\partial^2 F(x, t)}{\partial x\,\partial t} & \dfrac{\partial^2 F(x, t)}{\partial x^2} \end{vmatrix}=$$

$$=\begin{vmatrix} g'(f^t(x))g(f^t(x)) & \dfrac{g'(f^t(x))g(f^t(x))}{g(x)} \\[2ex] \dfrac{g'(f^t(x))g(f^t(x))}{g(x)} & \dfrac{g'(f^t(x))g(f^t(x))-g(f^t(x))g'(x)}{g(x)^2} \end{vmatrix}=$$

$$=\frac{-g'(f^t(x))g'(x)g(f^t(x))^2}{g(x)^2}\leqq 0,$$

since $g'(x)\leqq 0$ in J as a derivative of decreasing function.

Remark 4. If $\{f^t, t\in(-\infty, \infty)\}$ is a convex (concave) iteration group and there exists the derivative $\left.\dfrac{\partial f^t(x)}{\partial t}\right|_{t=0}=g(x)$ in J such that $g'(0)\neq 0$, then $\{f^t, t\in(-\infty, \infty)\}$ is of class C^1_*.

In fact, from Theorem 9 it follows that g is convex (concave), whence g is continuous, $g(x)<0$ for $x\in(0, a)$ and there exists the derivative $g'(0)\neq 0$. Now, our assertion results from Theorem 4.

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