Some problems of characterization of normal distribution

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1. Introduction

Consider two random variables ξ and η and assume that the conditional expectation $E(\eta|\xi)$ exists. We say that η has polynomial regression of order k on ξ if the relation

$$E(\eta|\xi) = \beta_0+\beta_1\xi+\ldots+\beta_k\xi^k \tag{1.1}$$

holds almost everywhere. Suppose that the first moment of η and the k-th moment of ξ exist. It follows from (1.1) that

$$E(\eta) = \beta_0+\beta_1 E(\xi)+\ldots+\beta_k E(\xi^k). \tag{1.2}$$

The coefficients $\beta_0, \beta_1, \ldots, \beta_k$ are called the *regression coefficients*. If $k=0$, i.e. if the relation $E(\eta/\xi)=E(\eta)$ holds almost everywhere, then we say that η has *constant regression on* ξ. If $k=1$ and $\beta_1\neq 0$ ($k=2$ and $\beta_2\neq 0$) then we speak of *linear (quadratic) regression*.

In this paper we discusse some characterization problems of normal distribution. They are connected with constant, linear and quadratic regression.

We shall need the following theorem.

Theorem 1.1. ([2], p. 103, Theorem 6.1.1.) *Let ξ and η be two random variables and suppose that the expectations $E(\eta)$ and $E(\xi^k)$ exists where k is a nonnegative integer. The random variable η has polynomial regression of order k on ξ if, and only if, the relation*

$$E(\eta e^{it\xi}) = \sum_{j=0}^{k}\beta_j E(\xi^j e^{it\xi}) \tag{1.3}$$

holds for all real t.

2. Characterization of normal distribution by constant regression of a quadratic statistic on a linear one

Let R_n be the n-dimensional vector (column) space. If $v\in R_n$, then v^* stands for the transpose of v. $a\in R_n$ is the vector with all components equal to 1.

Let the components of the random vector $\zeta=(\xi_j)\in R_n$ be independent, identically distributed random variables. Let

$$\Lambda = \sum_{i=1}^{n}\alpha_i\xi_i, \quad \alpha=(\alpha_i)\in R_n, \quad \alpha_i\neq 0, \quad i=1,\ldots,n \tag{2.1}$$

be a linear,

$$(2.2)\qquad Q=\sum_{i,j=1}^{n}a_{ij}\xi_i\xi_j+\sum_{k=1}^{n}b_k\xi_k,\quad a_{ij}\in R_1,\quad b_j\in R_1,\quad i,j=1,\dots,n$$

be a quadratic statistics. We prove now the following theorem.

Theorem 2.1. *Let the distribution function of ξ_j be $F(x)$ and assume that $F(x)$ has a finite second moment. Put*

$$\alpha=(\alpha_j)\in R_n,\quad \alpha_j\neq 0,\quad b=(b_j)\in R_n,\quad A=(a_{jk}),\quad a_{jk}\in R_1,\quad j,k=1,\dots,n.$$

Suppose that

$$(2.3)\qquad (xa+y\alpha)^*A(xa+y\alpha)\equiv 0,$$

$$(2.4)\qquad S_\nu=a_{11}\alpha_1^\nu+\dots+a_{nn}\alpha_n^\nu\neq 0,\quad \nu=1,2,\dots,$$

$$(2.5)\qquad S_0>0$$

$$(2.6)\qquad P_\mu=b_1\alpha_1^\mu+\dots+b_n\alpha_n^\mu=0,\quad \mu=0,1.$$

Then (2.2) *has constant regression on* (2.1) *if, and only if $F(x)$ is the normal distribution function.*

First we prove the following lemma.

Lemma 2.1. *Denote the characteristic function of $F(x)$ by $f(t)$. If* (2.2) *has constant regression on* (2.1), *then $f(t)$ satisfies the differential equation*

$$(2.7)\qquad \begin{aligned}&\sum_{j=1}^{n}a_{jj}f''(\alpha_jt)\prod_{\substack{k=1\\k\neq j}}^{n}f(\alpha_kt)-\sum_{\substack{j,k=1\\j\neq k}}^{n}a_{jk}f'(\alpha_jt)f'(\alpha_kt)\prod_{\substack{l=1\\l\neq j,l\neq k}}^{n}f(\alpha_lt)-\\&-i\sum_{j=1}^{n}b_jf'(\alpha_jt)\prod_{\substack{k=1\\k\neq j}}^{n}f(\alpha_kt)=\prod_{k=1}^{n}f(\alpha_kt)(B_1\sigma^2+B_2m^2+B_3m).\end{aligned}$$

Moreover, in a neighbourhood of the origin, in which $f(\alpha_jt)\neq 0$ $(j=1,2,\dots,n)$, the second characteristic $\varphi(t)=\ln f(t)$ satisfies the differential equation

$$(2.8)\qquad -\sum_{j=1}^{n}a_{jj}\varphi''(\alpha_jt)-\sum_{j,k=1}^{n}a_{jk}\varphi'(\alpha_jt)\varphi'(\alpha_kt)-i\sum_{j=1}^{n}b_j\varphi'(\alpha_jt)=B_1\sigma^2+B_2m^2+B_3m$$

where

$$B_1=\sum_{j=1}^{n}a_{jj},\quad B_2=\sum_{j,k=1}^{n}a_{jk},\quad B_3=\sum_{j=1}^{n}b_j.$$

Proof *of lemma 2.1.* As (2.2) has constant regression on (2.1)

$$E(Qe^{it\Lambda})=E(Q)E(e^{it\Lambda})$$

holds for all real $t\in R_1$. Since ξ_j has a finite second moment, we may differentiate the characteristic function $f(t)$ twice and get

$$f'(t) = iE(\xi_j e^{it\xi_j}) \tag{2.10}$$

$$f''(t) = -E(\xi_j^2 e^{it\xi_j}). \tag{2.11}$$

From these we obtain

$$E(Qe^{it\Lambda}) = -\sum_{j=1}^{n} a_{jj} f''(\alpha_j t) \prod_{\substack{k=1 \\ k\neq j}}^{n} f(\alpha_k t) -$$
$$- \sum_{\substack{j,k=1 \\ j\neq k}}^{n} a_{jk} f'(\alpha_j t) f'(\alpha_k t) \prod_{\substack{l=1 \\ l\neq j, l\neq k}}^{n} f(\alpha_l t) - i \sum_{j=1}^{n} b_j f'(\alpha_j t) \prod_{\substack{k=1 \\ k\neq j}}^{n} f(\alpha_k t). \tag{2.12}$$

Upon replacements and the equations

$$E(e^{it\Lambda}) = \prod_{j=1}^{n} f(\alpha_j t)$$
$$E(Q) = \sum_{j=1}^{n} a_{jj}\sigma^2 + \sum_{j,k=1}^{n} a_{jk} m^2 + m \sum_{j=1}^{n} b_j \tag{2.13}$$

we obtain the differential equation (2.7).

Moreover, in a neighbourhood of the origin, in which $f(\alpha_j t) \neq 0$ $(j=1, 2, \ldots, n)$, let us divide both sides of (2.7) by $\prod_{j=1}^{n} f(\alpha_j t)$ and introduce $\varphi(t) = \ln f(t)$. Then we obtain the differential equation (2.8) from (2.7).

PROOF *of theorem 2.1.* It follows from (2.3), (2.4) and

$$n \sum_{i=1}^{n} \xi_i^2 - \sum_{k,l=1}^{n} \xi_k \xi_l \geqq 0$$

that

$$Q^* = \frac{1}{n!} \sum_{(k_1, \ldots, k_n) \in P_n} \left[\sum_{i,j=1}^{n} a_i \xi_{k_i} \xi_{k_j} + \sum_{j=1}^{n} b_j \xi_{k_j} \right]$$

(the adjoint polynomial statistic of Q) is nonnegative. Then ([3], Theorem 4) the characteristic function $f(t)$ is an entire function and $\varphi(\alpha_j t)$ $(j=1, \ldots, n)$ is an analytic function in the interval $|\alpha_j t| < \delta$ $(\delta > 0)$. This indicates that $\varphi(\alpha_j t)$ satisfies the differential equation (2.8). If we put in (2.8) $t=0$ and use (2.3), (2.4), (2.5), we obtain

$$\varphi''(0) = -\sigma^2. \tag{2.14}$$

Now we show that

$$\varphi^{(s)}(0) = 0 \tag{2.15}$$

for all $s \geqq 3$. Differentiating both sides of (2.8) and then putting $t=0$ we have

$$\varphi'''(0) \sum_{j=1}^{n} a_{jj}\alpha_j = \varphi'(0)\varphi''(0) \left[\sum_{j,k=1}^{n} a_{jk}\alpha_j + \sum_{j,k=1}^{n} a_{jk}\alpha_k \right] - \varphi''(0) \cdot i \sum_{j=1}^{n} b_j \alpha_j. \tag{2.16}$$

Replacing the relations (2.3), (2.4), (2.6) into (2.16) we see easily that

$$\varphi'''(0) = 0. \tag{2.17}$$

Now we differentiate both sides of (2.8) twice and then put $t=0$ and use the relations (2.3), (2.4) and (2.17), we get

$$\varphi''''(0) = 0. \tag{2.18}$$

We prove (2.15) by induction. Let us suppose that (2.15) holds for $s=3, 4, \ldots, \nu+1$. We show that (2.15) is valid for $s=\nu+2$.

Differentiate both sides of (2.8) ν times $(\nu\geqq 3)$ and then put $t=0$. We have that

(2.19)

$$\varphi^{(\nu+2)}(0)\cdot S_\nu = -\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=0}^{\nu}\binom{\nu}{l}\varphi^{(l+1)}(0)\varphi^{(\nu+1-l)}(0)\alpha_j^l\alpha_k^{\nu-l} - \varphi^{(\nu+1)}(0)\cdot i\sum_{j=1}^{n} b_j\alpha_j^{\nu+1}.$$

Since $\nu\geqq 3$, $\max\{l+1, \nu+1-l\}\geqq 3$. Therefore the right-hand side of (2.19) is equal to zero. We use (2.3) and get

$$\varphi^{(\nu+2)}(0) = 0.$$

This completes the proof of (2.15).

Since $\varphi(\alpha_j t)$ $(j=1, 2, \ldots, n)$ is an analytic function in the interval $|\alpha_j t|<\delta$, therefore in this interval the equation

$$\varphi(t) = -\frac{\sigma^2 t^2}{2} + at + b,$$

where a and b are complex constants is valid. As $\varphi(0)=0$ and $\varphi(-t)=\overline{\varphi(t)}$ thus $b=0$, $a=\mathrm{im}\ m\in R_1$ and

$$\varphi(t) = -\frac{\sigma^2 t^2}{2} + \mathrm{im}\ t$$

Since $f(t)$ is an entire characteristic function, the equation

$$f(t) = \exp\left\{-\frac{\sigma^2 t^2}{2} + \mathrm{im}\ t\right\}$$

is valid for $t\in R_1$ and so ξ_j is normally distributed.

On the other hand the characteristic function of the normal distribution satisfies the differential equation (2.7). We put in (2.7) equations (2.12) and (2.13) and obtain that (2.2) has constant regression on (2.1).

3. A characterization of normal distribution by linear regression of quadratic statistics on linear one

Let the components of the random vector $\zeta=(\xi_j)\in R_n$ be independent random variables with common distribution function $F(x)$ which has moments of all orders. Let m be expectation and let σ be the variance of the distribution $F(x)$.

Theorem 3.1. *Let*

$$\alpha=(\alpha_j)\in R_n,\quad \alpha_j\neq 0,\quad b=(b_j)\in R_n,\quad A=(a_{jk}),\quad a_{jk}\in R_1,\quad j,k=1,\dots,n.$$

Suppose that

$$(xa+y\alpha)^*A(xa+y\alpha)\equiv 0, \tag{3.1}$$

$$S_\nu=a_{11}\alpha_1^\nu+\dots+a_{nn}\alpha_n^\nu\neq 0,\quad |\nu=1,2,\dots|, \tag{3.2}$$

$$S_0>0, \tag{3.3}$$

$$\sum_{j=1}^n(\beta_1\alpha_j-b_j)=0,\quad \sum_{j=1}^n b_j=0, \tag{3.4}$$

$$\sum_{j=1}^n(\beta_1\alpha_j-b_j)\alpha_j=0, \tag{3.5}$$

$$\sigma^2=\frac{\beta_0}{\sum\limits_{i=1}^n a_{ii}}. \tag{3.6}$$

Then (2.2) *has linear regression on* (2.1) *if, and only if $F(x)$ is the normal distribution.*

The proof will be based on the following lemma.

Lemma 3.1. *First let us denote again by $f(t)$ the characteristic function of $F(x)$ and suppose that $F(x)$ has a finite second moment. If* (2.2) *has linear regression on* (2.1) *then $f(t)$ satisfies the differential equation*

$$\begin{aligned}-\sum_{j=1}^n a_{jj}f''(\alpha_jt)\prod_{\substack{k=1\\k\neq j}}^n f(\alpha_kt)-\sum_{\substack{j,k=1\\j\neq k}}^n a_{jk}f'(\alpha_kt)f'(\alpha_jt)\prod_{\substack{l=1\\l\neq j,l\neq k}}^n f(\alpha_lt)+\\+i\sum_{j=1}^n(\beta_1\alpha_j-b_j)f'(\alpha_jt)\prod_{\substack{k=1\\k\neq j}}^n f(\alpha_kt)=\beta_0\prod_{j=1}^n f(\alpha_jt).\end{aligned} \tag{3.7}$$

Moreover, in a neighbourhood of the origin, in which $f(\alpha_jt)\neq 0$ $(j=1,2,\dots,n)$, the second characteristic $\varphi(t)=\ln f(t)$ satisfies the differential equation

$$-\sum_{j=1}^n a_{jj}\varphi''(\alpha_jt)-\sum_{j,k=1}^n a_{jk}\varphi'(\alpha_jt)\varphi'(\alpha_kt)+i\sum_{j=1}^n(\beta_1\alpha_j-b_j)\varphi'(\alpha_jt)=\beta_0. \tag{3.8}$$

PROOF *of lemma 3.1.* If (2.2) has linear regression on (2.1) then

$$E(Qe^{it\Lambda})=\beta_0E(e^{it\Lambda})+\beta_1E(\Lambda e^{it\Lambda}) \tag{3.9}$$

holds for all real $t \in R_1$. Since $F(x)$ has a finite second moment we may differentiate the characteristic function $f(t)$ twice and thus (2.10) and (2.11) are valid. We use (2.12) and the equations

$$E(\Lambda e^{it\Lambda}) = -\sum_{j=1}^{n} \alpha_j f'(\alpha_j t) \prod_{\substack{k=1 \\ k \neq j}}^{n} f(\alpha_k t), \tag{3.10}$$

$$E(e^{it\Lambda}) = \prod_{j=1}^{n} f(\alpha_j t) \tag{3.11}$$

and get the differential equation (3.7).

In a neighbourhood of the origin, in which $f(\alpha_j t) \neq 0$ $(j=1, 2, \ldots, n)$ let us divide both sides of (3.7) by $\prod_{j=1}^{n} f(\alpha_j t)$ and introduce $\varphi(t) = \ln f(t)$. Then (3.8) follows from (3.7).

PROOF *of theorem 3.1.* The proof is similar to the proof of theorem 2.1. It is easy to see that Q^* is a nonnegative polynomial statistic. Then ([1], Theorem 4) the characteristic function $f(t)$ is an entire function and $\varphi(\alpha_j t)$ $(j=1, 2, \ldots, n)$ is an analytic function in the interval $|\alpha_j t| < \delta$ $(\delta > 0)$. This indicated that $\varphi(\alpha_j t)$ satisfies (3.8).

Similarly to the proof of theorem 2.1. we obtain that

$$\varphi''(0) = -\sigma^2 \tag{3.12}$$

and

$$\varphi^{(s)}(0) = 0, \quad s = 3, 4, \ldots . \tag{3.13}$$

Since $\varphi(\alpha_j t)$ $(j=1, 2, \ldots, n)$ is an analytic function in the interval $|\alpha_j t| < \delta$, therefore in this interval

$$\varphi(t) = \frac{-\sigma^2 t^2}{2} + \mathrm{im}\, t. \tag{3.14}$$

Since $f(t)$ is an entire characteristic function it should have the form

$$f(t) = \exp\left\{\frac{-\sigma^2 t^2}{2} + \mathrm{im}\, t\right\} \tag{3.15}$$

for all $t \in R_1$. Therefore $F(x)$ is the normal distribution.

The final steps are analogous to that of theorem 2.1.

4. Characterization by quadratic regression of a quadratic statistic on a linear one

We shall keep the notations and suppositions introduced in section 3. Additionally assume that $f(t)$ is an entire characteristic function.

Theorem 4.1. *Let*

$$\alpha = (\alpha_j) \in R_n, \quad \alpha_j \neq 0, \quad b = (b_j) \in R_n, \quad A = (a_{jk}), \quad a_{jk} \in R_1, \quad j, k = 1, \ldots, n.$$

Suppose that

$$R_\nu = \sum_{j=1}^{n} (\beta_2 \alpha_j^2 - a_{jj}) \alpha_j^\nu \neq 0, \quad \nu = 0, 1, 2, \ldots, \tag{4.1}$$

$$\sum_{j,k=1}^{n} (\beta_2 \alpha_j \alpha_k - a_{jk}) = 0, \tag{4.2}$$

$$\sum_{j,k=1}^{n} (\beta_2 \alpha_j \alpha_k - a_{jk})(\alpha_j + \alpha_k) = 0, \tag{4.3}$$

$$\sum_{j,k=1}^{n} (\beta_2 \alpha_j \alpha_k - a_{jk}) \alpha_j \alpha_k = 0, \tag{4.4}$$

$$\sum_{j=1}^{n} (\beta_1 \alpha_j - b_j) = 0, \tag{4.5}$$

$$\sum_{j=1}^{n} (\beta_1 \alpha_j - b_j) \alpha_j = 0, \tag{4.6}$$

$$\sigma^2 = \frac{\beta_0}{\sum_{j=1}^{n} (a_{jj} - \beta_2 \alpha_j^2)}. \tag{4.7}$$

Then (2.2) *has quadratic regression on* (2.1) *if, and only if* $F(x)$ *is the normal distribution function.*

For the proof of theorem 4.1. we need the following lemma.

Lemma 4.1. *If* (2.2) *has quadratic regression on* (2.1), *then the characteristic function* $f(t)$ *satisfies the differential equation*

$$\begin{gathered} \sum_{j=1}^{n} (\beta_2 \alpha_j^2 - a_{jj}) f''(\alpha_j t) \prod_{\substack{k=1 \\ k \neq j}}^{n} f(\alpha_k t) + \sum_{\substack{j,k=1 \\ j \neq k}}^{n} (\beta_2 \alpha_j \alpha_k - a_{jk}) \cdot f'(\alpha_k t) f'(\alpha_j t) \prod_{\substack{l=1 \\ l \neq j,\, l \neq k}}^{n} f(\alpha_l t) + \\ + i \sum_{j=1}^{n} (\beta_1 \alpha_j - b_j) f'(\alpha_j t) \prod_{\substack{k=1 \\ k \neq j}}^{n} f(\alpha_k t) = \beta_0 \prod_{k=1}^{n} f(\alpha_k t). \end{gathered} \tag{4.8}$$

Moreover, in a neighbourhood of the origin, in which $f(\alpha_j t) \neq 0$ $(j = 1, 2, \ldots, n)$, *the second characteristic* $\varphi(t) = \ln f(t)$ *satisfies the differential equation*

$$\begin{gathered} \sum_{j=1}^{n} (\beta_2 \alpha_j^2 - a_{jj}) \varphi''(\alpha_j t) + \sum_{j,k=1}^{n} (\beta_2 \alpha_j \alpha_k - a_{jk}) \varphi'(\alpha_j t) \varphi'(\alpha_k t) + \\ + i \sum_{j=1}^{n} (\beta_1 \alpha_j - b_j) \varphi'(\alpha_j t) = \beta_0. \end{gathered} \tag{4.9}$$

PROOF *of lemma 4.1.* If (2.2) has quadratic regression on (2.1) then

$$E(Q e^{it\Lambda}) = \beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda}) \tag{4.10}$$

holds for all real $t \in R_1$. If we replace (2.12), (3.10), (3.11) and the equation

$$(4.11)\quad E(\Lambda^2 e^{it\Lambda}) = -\sum_{j=1}^{n} \alpha_j^2 f''(\alpha_j t) \prod_{\substack{k=1 \\ k \neq j}}^{n} f(\alpha_k t) - \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \alpha_j \alpha_k f'(\alpha_j t) f'(\alpha_k t) \prod_{\substack{l=1 \\ l \neq j, l \neq k}}^{n} f(\alpha_l t)$$

into (4.10) we obtain the differential equation (4.8). Moreover, in a neighbourhood of the origin, in which $f(\alpha_j t) \neq 0$ $(j=1, 2, \dots, n)$ if we divide both sides of (4.8) by $\prod_{j=1}^{n} f(\alpha_j t)$ and introduce $\varphi(t) = \ln f(t)$ then (4.8) yields (4.9).

Proof *of theorem 4.1.* Similarly to the proof of theorem 2.1. we establish that

$$(4.12)\qquad \varphi''(0) = -\sigma^2$$

and

$$(4.13)\qquad \varphi^{(s)}(0) = 0, \quad s = 3, 4, \dots .$$

Since $\varphi(\alpha_j t)$ $(j=1, 2, \dots, n)$ is an analytic function in the interval $|\alpha_j t| < \delta$, therefore in this interval

$$(4.14)\qquad \varphi(t) = \frac{-\sigma^2 t^2}{2} + \mathrm{i}m\, t.$$

As $f(t)$ is supposed to be an entire characteristic function, the equation

$$(4.15)\qquad f(t) = \exp\left\{\frac{-\sigma^2 t^2}{2} + \mathrm{i}m\, t\right\}$$

holds for all $t \in R_1$, i.e. $F(x)$ is the normal distribution.

References

[1] B. Glevitzky, On an extension of a theorem of Linnik and Zinger *Publ. Math. (Debrecen)* **24** (1977), 151—161.

[2] E. Lukács—R. G. Laha, Applications of characteristic functions, *New York*, 1964.

[3] B. Gyires, On an extension of a theorem of Linnik and Zinger (*Advances in Applied Probability, Vol.* **7** pp. 786—800.)

[4] B. Gyires, A characterization of the normal distribution. (*Statistical distribution in Scientific Work. Vol.* **3**. *Boston.* 211—219.)

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