Notes on matrix-valued stationary stochastic processes II

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1. Introduction

We are going to construct the prediction of the matrix-valued stationary processes (m. st. p.) with lag 1 and more. For this reason the connection of the matrix-valued moving-average processes with the spectral density functions and the Q. H. space spanned by the m. st. p. will be examined. The definitions and the designations of the first part will be used, thus, the m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$, a curve in the Q. H. space $\mathcal{M}_2^{p\times q}$, can denote a vector, a quadratic matrix, a rectangle matrix and a ribbon matrix-valued discrete processes.

2. Moving-average process and spectral density

As in the vector case there is a close connection between the representation of the m. st. p. by moving average and the integrability of the logarithm of the determinant (det) of the spectral density function. In the case where the spectral distribution function $F(\cdot)$ is absolutely continuous we will either call the matrix function

$$f(\lambda) = F'(\lambda) = [F'_{kl}(\lambda)]_{k,l=1}^{p},$$

the spectral density function of the m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$ or say that the spectral density function exists.

The sets $\mathfrak{M}_{p,\delta}\,(\delta>0)$ contain all M_p valued functions $F(e^{i\Theta})=[F_{kl}(e^{i\Theta})]_{k,l=1}^{p}$ on the unit circle $(\Theta\in[0, 2\pi])$ with complex-valued entires $F_{kl}(\cdot)$ for which

$$\int_0^{2\pi} |F_{kl}(e^{i\Theta})|^{\delta}\, d\Theta < \infty$$

is fulfilled.

If $F(\cdot)\in\mathfrak{M}_{p,\delta}$ then $F_+(\cdot)$ denotes the function

$$F_+(z) = \sum_{n=0}^{\infty} A_n z^n$$

on the set $|z|<1$ of the complex plane, where

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\Theta})e^{in\Theta}\, d\Theta \quad (\in M_p)$$

the n-th Fourier coefficient of $F(\cdot)$. As the proofs of the following two theorems come readily from the corresponding ones for the vector-valued stationary processes, i.e., theorems of WIENER—MASANI ([5] 3.13. and 7.13. Theorem) and from the methods of Q. H. space, we shall omit them.

2.1. Theorem. *If* $\{\xi_n\}_{-\infty}^{\infty}$ *is moving-average process, i.e.,*

$$\xi_n = \sum_{k=0}^{\infty} A_k \eta_{n-k}, \quad A_k \in M_p. \tag{2.1}$$

Where $\eta_k \in \mathcal{M}_2^{p\times q}$, $\langle \eta_k, \eta_l \rangle = \delta_{k_l} G$ *and* $\sum_{k=0}^{\infty} sp(A_k G A_k^*) < \infty$ *(sp refers to the trace of matrices) then*

a) $\{\xi_n\}_{-\infty}^{\infty}$ *is a m. st. p.;*

b) *the spectral density function* $f(\cdot)$ *exists and*

$$f(e^{i\Theta}) = \Phi(e^{i\Theta})\,\Phi^*(e^{i\Theta})$$

where

$$\Phi(\cdot) \in \mathcal{M}_{p,2}, \quad \Phi(e^{i\Theta}) = \sum_{k=0}^{\infty} A_k G^{1/2} e^{-ik\Theta}, \quad \Theta \in [0, 2\pi];$$

c) *either* $\det \Phi_t(z) \equiv 0$ *or*

$$\log\det(A_0 G A_0^*) \leqq \frac{1}{2\pi} \int_0^{2\pi} \log\det f(e^{i\Theta})\, d\Theta.$$

2.2. Theorem. *If* $\{\xi_n\}_{-\infty}^{\infty}$ *is a m. st. p. and the spectral density function* $f(\cdot)$ *exists and* $f(\cdot) \in \mathfrak{M}_{p,1}$ *and*

$$\int_0^{2\pi} \log\det f(e^{i\Theta})\, d\Theta > -\infty \tag{2.2}$$

then

a) $\{\xi_n\}_{-\infty}^{\infty}$ *is a moving average process, i.e.,*

$$\xi_n = \sum_{k=0}^{\infty} A_k \eta_{n-k}, \quad A_k \in M_p$$

where

$$\eta_k \in \mathcal{M}_2^{p\times q}, \quad \langle \eta_k, \eta_l \rangle = \delta_{kl} I \sum_{k=0}^{\infty} \mathrm{sp}\,(A_k A_k^*) < \infty$$

($I \in M_p$ is the unit matrix).

b)

$$\det(A_0 A_0^*) = \exp \frac{1}{2\pi} \int_0^{2\pi} \log\det f(e^{i\Theta})\, d\Theta. \tag{2.3}$$

3. Q. H. space of a m. st. p.

The Q. H. subspace of $\mathcal{M}_2^{p\times q}$ spanned by the set of the m. st. p. $\{\xi_k\}_{-\infty}^{n}$ will be denoted by $\sigma(\xi_k|_{-\infty}^{n})$ or G_n simple. It is clear that

$$\mathcal{M}_2^{p\times q} \supseteqq \sigma_\infty \supseteqq \sigma_n$$

and they are subspaces. If $\tau \in \mathcal{M}_2^{p\times q}$ but $\tau \notin \sigma_n$ then the orthogonal projection of τ on σ_n is the $[\tau|\sigma_n]$. For every $\zeta \in \sigma_n$ the $\tau-[\tau|\sigma_n]\perp\zeta$, i.e.,

$$\langle \tau-[\tau|\sigma_n], \zeta\rangle = 0 \quad (\in M_p, \text{ zero matrix}).$$

We shall say that the m. st. p. $\{\xi_n\}_{-\infty}^{\infty}$ is non-deterministic if for any n, $\xi_n \notin \sigma_{n-1}$. From the stationarity property (1.0) it follows that the last relation holds for *a* single n only if it holds for all n.

Let us define the innovation-process $\{\eta_k\}_{-\infty}^{\infty}$ of a non-deterministic m. st. p. $\{\zeta_n\}_{-\infty}^{\infty}$ by

$$\eta_k = \xi_k - [\xi_k|\sigma_{k-1}].$$

We can calculate

$$\langle \eta_n, \eta_m\rangle = \delta_{n,m} G$$

where

$$G = \langle \eta_0, \eta_0\rangle = \langle \xi_n, \eta_n\rangle.$$

The Q. H. space spanned by $\{\eta_k\}_{-\infty}^{n}$ will be denoted by $\sigma(\eta_k|_{-\infty}^{n})$ or σ_n.

3.1. Theorem. *If* $m<n$ *then the following statements hold*

a) $\sigma_n = \sigma_m + \sigma(\eta_k|_{m+1}^{n})$ and $\sigma_m \perp \sigma(\eta_k|_{m+1}^{n})$

b) $\sigma_n = \sigma_{-\infty} + \sigma^n$ and $\sigma_{-\infty} \perp \sigma_\infty$ where $\sigma_{-\infty} = \bigcap_{-\infty}^{\infty} \sigma_k$

c)

$$\sigma^n = \sum_{k=0}^{\infty} \sigma(\eta_{n-k}). \tag{3.1}$$

This theorem is provable by methods of Wiener—Masani [5] (6.10. Lemma) making use of the concept of Q. H. space.

In the next theorem three equivalent statements are summed up. One of them is an analytic hypothesis for the spectral density functions. The two others are the conditions of representation by moving averages and the "remote past" of the process.

3.2. Theorem. *If* $\{\xi_n\}_{-\infty}^{\infty}$ *is a m. st. p. then the following statements are equivalent:*

a) *the process is non-deterministic,* $G_{-\infty} = \{0\}$ *and* $\det G \neq 0$,

b) $\{\xi_n\}_{-\infty}^{\infty}$ *is a matrix valued moving average process and* $\det G \neq 0$,

c) *the spectral density function* $f(\cdot)$ *exists for the m. st. p.* $\{\xi_n\}_{-\infty}^{\infty}$, $f(\cdot) \in \mathfrak{M}_{p,1}$ *and*

$$\int_0^{2\pi} \log\det f(e^{i\Theta})\, d\Theta > -\infty.$$

Proof. It is sufficient to verify only that

$$\int_0^{2\pi} \log\det f(e^{i\Theta})\, d\Theta > -\infty$$

if $\det G \neq 0$. This follows from the 2.1. Theorem since

$$A_0 G = \langle \xi_0, \eta_0 \rangle = \langle \eta_0, \eta_0 \rangle = G = \langle \eta_0, \xi_0 \rangle = G A_0^*$$

so $A_0 = I$ and $\Phi_+(0) = G^{1/2}$.

4. Prediction of a m. st. p.

Let $\{\xi_n\}_{-\infty}^{\infty}$ be a non-deterministic m. st. p., $\{\eta_k\}_{-\infty}^{\infty}$ its innovation process, $\det \langle \eta_0, \eta_0 \rangle = \det \sigma \neq 0$ and $\sigma_{-\infty} = \{0\}$.

Thus

$$\xi_n = \sum_{k=0}^{\infty} A_k \eta_{n-k} \quad (A_0 = I). \tag{4.1}$$

We want to construct the prediction $\tilde{\xi}_k$ of ξ_k $(k > 0)$ when the $\{\xi_n\}_{-\infty}^{0}$ is known. The best linear predictor is

$$\tilde{\xi}_k = [\xi_k | \sigma_0] = [\xi_k | \sigma^0] = \sum_{l=k}^{\infty} A_l \eta_{k-l}. \tag{4.2}$$

The error matrix, by definition, is

$$\langle \xi_k - \tilde{\xi}_k, \xi_k - \tilde{\xi}_k \rangle = \sum_{l=0}^{k-1} A_l G A_l^*. \tag{4.3}$$

The error matrix of prediction with lag 1 is G and

$$\det G = \exp \frac{1}{2\pi} \int_0^{2\pi} \log \det f(e^{i\Theta})\, d\Theta, \tag{4.4}$$

where $f(\cdot)$ is the spectral density function of the process.

In discrete case we were able to announce some results which are comparable to vector valued stationary processes, for the m. st. p. which is a curve in. In my opinion there is no difficulty in the continuous case either. The importance of the research of the m. st. p. is greater from the standpoint of statistics as the definition of stationarity is weaker.

Bibliography

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