On cosine operator functions

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Introduction. Let X be a Banach space over the complex field, $B(X)$ the space of bounded linear operators from X into X, R the real field. A *cosine operator function* is a mapping $C: R \to B(X)$ such that $C(0)=I$ (the identical operator), for $s, t \in R$

$$C(s+t)+C(s-t) = 2C(s)\,C(t), \tag{1}$$

and $t \to t_0$ implies $C(t) \to C(t_0)$ in the strong operator topology of $B(X)$. For the basic facts on cosine operator functions see e.g. [1], [3] and [4].

Let A be a closed linear operator with domain $D(A)$ dense in X, range $R(A)$ n X and non-void resolvent set $\varrho(A)$. Consider the differential equation

$$u''(t) = Au(t), \tag{2}$$

where $u: R \to D(A)$. H. FATTORINI [1] has shown that the (generalized) solutions of (2) are closely connected with the cosine operator functions and their indefinite integrals, defined by $T(t)x = \int_0^t C(s)x\,ds$ $(x \in X)$, supposing the Cauchy problem for (2) is uniformly well posed (u.w.p.) in R.

The first part of this paper investigates the behaviour of cosine operator functions and their indefinite integrals at infinity. As a rule the limits $\lim_{t\to\infty} C(t)$ and $\lim_{t\to\infty} T(t)$ do not exist even in the uniformly bounded case in any of the standard operator topologies of $B(X)$, therefore, following [2] (Chap. 18), we employ the Cesàro (C_a) and the Abel limits. Some of the results can be applied to study the behaviour of the solutions of (2) under suitable conditions.

In the second part we prove a result characterizing the kernel of an operator $C(b)$ in the range of a cosine operator function.

1.

Definition 1. Suppose $F:(0, \infty) \to B(X)$ is a strongly measurable operator function such that for $z>0$, $x \in X$, $e^{-zt}F(t)x$ is Bochner integrable on $(0, \infty)$ relative to Lebesgue measure, and with the notation $A(z)x = z\int_0^\infty e^{-zt}F(t)x\,dt$ we have $A(z) \in B(X)$. We say that $F(t)$ is weakly (strongly, uniformly) Abel-convergent at infinity and its Abel limit is $P \in B(X)$, if $\lim_{z\to 0+} A(z) = P$ in the respective operator topologies of $B(X)$.

It is well-known that for every cosine operator function $C(t)$ there are numbers $w \geqq 0$, $M(w) \geqq 1$ such that on R $\|C(t)\| \leqq M(w)e^{w|t|}$. However, in general there is no minimal growth parameter w (cf. [4], Remark on p. 10.).

Definition 2. If $C(t)$ is a cosine operator function, we put $w_0 = \inf\{w \geqq 0;\ e^{-w|t|}\|C(t)\| \text{ is bounded on } R\}$, and call $C(t)$ of type w_0.

Theorem 1. *Suppose the cosine operator function $C(t)$ is of type 0, and for $x \in X$ put $T(t)x = \int_0^t C(s)x\,ds$ $(t \in R)$. Then the following statements are equivalent:*

1) *$T(t)$ is weakly Abel-convergent at infinity,*
2) *$C(t)$ and $T(t)$ are uniformly Abel-convergent at infinity with limits 0,*
3) *there are numbers $v, K > 0$ such that on the interval $(0, v)$ we have $\|zR(z^2; A\| \leqq K$ $(R(u; A)$ will denote the resolvent of the generator operator A of $C(t))$,*
4) *there is a $v > 0$ such that for every $x \in X$ the set $H(x) = \{zR(z^2; A)x;\ 0 < z < v\}$ is conditionally sequentially weakly compact.*

Proof. 2) clearly implies 1). If 1) holds and $x \in X$, $x^* \in X^*$, then

$$x^*zR(z^2; A)x = x^*z\int_0^\infty e^{-zt}T(t)x\,dt \to x^*Px \quad \text{if} \quad z \to 0+. \tag{3}$$

Now if $v > 0$ and $\{z_n\} \subset (0, v)$ is a sequence, then for some subsequence $\{z_{n_k}\}$ of $\{z_n\}$ we have $z_{n_k} \to z_0 \in [0, v]$. If $z_0 > 0$ then, by the analyticity of the resolvent for $\operatorname{Re} z > 0$, we obtain $z_{n_k}R(z_{n_k}^2; A)x \to z_0R(z_0^2; A)x$, while if $z_0 = 0$, then by (3) we get that $H(x)$ is conditionally sequentially weakly compact, i.e. 4) follows.

If 4) holds, then according to [2] (Theor. 2. 9. 1.) $H(x)$ is bounded, and the principle of uniform boundedness yields 3).

Finally, if 3) is satisfied, then on the interval $(0, v)$ we have $\|z^2R(z^2; A)\| \leqq K \cdot z$, thus $z \to 0+$ implies $zR(z; A) \to 0$ in the uniform operator topology of $B(X)$. By [2] (Theor. 18. 8. 1.), $R(z; A)$ is then holomorphic in a neighbourhood of 0, hence in some neighbourhood of 0 $\|R(z^2; A)\|$ is bounded, thus $\|zR(z^2; A)\| \leqq N|z|$. Consequently $\lim_{z \to 0} zR(z^2; A) = 0$ in the uniform operator topology and, since $z\int_0^\infty e^{-zt}C(t)x\,dt = z^2R(z^2; A)x$ $(z > 0,\ x \in X)$, therefore 2) is true and the proof is complete.

Corollary. *Suppose the Cauchy problem for (2) is u.w.p. in R and of type $\leqq w$ for every $w > 0$. Then every solution of (2) is Abel-convergent at infinity if and only if 3) or 4) of Theorem 1 is valid. Moreover, then every generalized solution of (2) is Abel-convergent at infinity to 0.*

Remark. The Abel convergence of a function $f: (0, \infty) \to X$ has been defined in [2] (Sec. 18. 2.).

Proof. Under the given conditions the operator A generates a cosine operator function $C(t)$ of type 0 ([1] (5. 9. Theorem). If every solution is Abel-convergent, then for $x \in D(A)$

$$zR(z^2; A)x = z\int_0^\infty e^{-zt}T(t)x\,dt \to Px \quad (z \to 0+).$$

Moreover, then $zR(z^2; A)Ax = z(z^2 R(z^2; A)x - x) \to 0$, and if $y \in X$, $u \in \varrho(A)$, then $x_0 = R(u; A)y \in D(A)$, hence $zR(z^2; A)y = zR(z^2; A)(uI - A)x_0$ converges if $z \to 0+$. By the uniform boundedness principle, 3) and then 4) of Theorem 1 is valid. The remaining parts of the Corollary are evident, by Theorem 1.

In what follows $R(B)$ and $Z(B)$ denote the range and the zero subspace, respectively of a linear operator B, and $\overline{H}$ denotes the strong closure of the set H.

Theorem 2. *If $C(t)$ is a cosine operator function of type 0, then the following conditions are equivalent:*

1) *$C(t)$ is weakly Abel-convergent at infinity,*
2) *$C(t)$ is strongly Abel-convergent at infinity,*
3) *for some $v>0$ and for every $x \in X$ the set $\{zR(z; A)x;\ 0<z<v\}$ is conditionally sequentially weakly compact,*
4) *for some $v, K>0$, $0<z<v$ implies $\|zR(z; A)\| \leq K$, and $X = \overline{Z(A)+R(A)}$.*

Moreover, then the strong Abel limit is a projection operator $P \in B(X)$, for which

(i) $PC(t) = C(t)P = P$ *for* $t \in R$,

(ii) $APx = 0$ *for* $x \in X$, $PAx = 0$ *for* $x \in D(A)$,

(iii) $R(P) = Z(A) = \{x \in X;\ C(s)x = x \text{ for } s \in R\}$,

(iv) $Z(P) = \overline{R(A)}$,

(v) $X = \overline{R(A)} \oplus Z(A)$.

PROOF. If $C(t)$ is of type 0, then for every $z>0$, $x \in X$ we have

$$A(z)x = z \int_0^\infty e^{-zt} C(t)x\,dt = z^2 R(z^2; A)x,$$

and

$$\lim_{z \to 0+} A(z) = \lim_{z \to 0+} zR(z; A) \tag{4}$$

in the respective topologies. Moreover, then A generates a semigroup of operators $F(t)$, $t \in (0, \infty)$ of class (C_0) and of type ≤ 0 (cf. [1], 5. 11. Remark on p. 92). The Abel-convergence of $F(t)$ is equivalent to the existence of the limit (4). Thus the assertions of the theorem follow from [2], Secs. 18.5.—18.7., while (i) can be proved similarly.

Definition 3. Suppose the operator function $F(t)$ satisfies the conditions occurring in Definition 1, and that for $t, a>0$, $x \in X$ with the notation

$$C(t, a)x = at^{-a} \int_0^t (t-s)^{a-1} F(s)x\,ds$$

we have $C(t, a) \in B(X)$. We say that $F(t)$ is weakly (strongly, uniformly) C_a-convergent at infinity and its C_a limit is $Q \in B(X)$, if $\lim_{t \to \infty} C(t, a) = Q$ in the respective operator topologies of $B(X)$.

Theorem 3. *If $C(t)$ is a cosine operator function, for which on R $\|C(t)\| \leqq M$, then the following assertions are equivalent:*
1) *$C(t)$ is weakly C_a-convergent at infinity for some $a>0$,*
2) *$C(t)$ is strongly C_a-convergent at infinity for each $a>0$,*
3) *$X = \overline{Z(A) + R(A)}$.*

Moreover, then the C_a limit is a projection operator $P \in B(X)$ satisfying the assertions of the previous theorem.

PROOF. Under the given conditions for $z>0$ we have $\|zR(z^2; A)\| \leqq \dfrac{M}{z}$, thus

$$\|zR(z; A)\| \leqq M. \tag{5}$$

If 1) holds, then by [2] (Theorem 18. 2. 1.) $C(t)$ is weakly Abel-convergent and, according to Theorem 2, 3) is true. If 3) is valid, then (5) gives that $C(t)$ is strongly Abel-convergent and, by [2] (Theorem 18.3.3.) 2) holds, while 2) evidently implies 1).

Theorem 4. *Suppose that, with the notations of Theorem* 1, $\sup \{\|C(t)\|, \|T(t)\|; t \in R\} = M$ *is finite. Then for each $a>0$ $C(t)$ and $T(t)$ are strongly C_a-convergent at infinity to* 0.

PROOF. Since $R(z^2; A)x = \int\limits_0^\infty e^{-zt}T(t)x$ $(z>0,\ x \in X)$, thus $\|zR(z^2; A)\| \leqq M$ for $z>0$. By Theorem 1, $C(t)$ and $T(t)$ are strongly Abel-convergent at infinity to 0, and [2] (Theorem 18.3.3.) gives the assertion.

Theorem 5. *If the cosine operator function $C(t)$ is continuous with respect to the uniform operator topology of $B(X)$, and $\|C(t)\| \leqq M$ on R, then the following statements are equivalent:*
1) *$C(t)$ is uniformly Abel-convergent at infinity to $P \in B(X)$,*
2) *$C(t)$ is uniformly C_a-convergent at infinity to P for each $a>0$,*
3) *$z=0$ is a simple pole of $R(z; A)$ with residue P.*

PROOF. By our previous remarks, it follows from [2], Theorems 18.2.1., 18.3.3. and 18.8.4.

2.

It is known from the spectral theory of cosine operator functions [3] that 0 is an eigenvalue of the operator $C(b)$ $(b \neq 0)$ if and only if at least one element of $\left\{-\left[\frac{\pi}{b}\left(k+\frac{1}{2}\right)\right]^2;\ k \text{ integer}\right\}$ is an eigenvalue of A. The following theorem characterizes the kernel of $C(b)$.

Theorem 6. *$x \in X$ is in the kernel of the operator $C(b)$ $(b \neq 0)$ in the range of a cosine operator function if and only if the function*

$$f(z) = (1 + e^{-2|b|z})zR(z^2; A)x \quad \left\{z \in \sqrt{\varrho(A)}\right\}$$

can be analytically continued to an entire function $h(z)$ for which on the complex plane Z we have $\|h(z)\| \leqq Me^{2|b \cdot \mathrm{Re}\, z|}$.

PROOF. We may and will assume $x \neq 0$, further that $b > 0$, for $C(t)$ is an even function. To prove the only if part, put $X_0 = \{x \in X;\ C(b)x = 0\}$. Then X_0 is a non-trivial closed subspace, for which $C(s)X_0 \subset X_0$ for $s \in R$. Hence if A is the generator of $C(t)$, then $X_A = X_0 \cap D(A)$ is dense in X_0, and the restriction of A to X_A is closed with $AX_A \subset X_0$. Suppose $z^2 \in \varrho(A)$ and $x \in X$, then $C(b)R(z^2; A)x = R(z^2; A)C(b)x$, hence $R(z^2; A)X_0 \subset X_A$, and the restriction of $R(z^2; A)$ to X_0 is the unique bounded linear operator in X_0 for which

$$R(z^2; A)(z^2 I - A)x = x \quad (x \in X_A),$$

$$(z^2 I - A)R(z^2; A)x = x \quad (x \in X_0).$$

Put $z_k = (2k+1)\dfrac{i\pi}{2b}$ (k integer), $H = \{z_k;\ k \text{ integer}\} \cup \{0\}$, $z \in Z \setminus H$, $x \in X_0$ and

$$U(z)x = z^{-1}(1+e^{-2bz})^{-1} \int_0^{2b} e^{-zu} C(u)x\, du.$$

Then $U(z)$ is a bounded linear operator in X_0, and $U(z)Ax = AU(z)x$ for $x \in X_A$. $x \in X_0$ implies

$$C(s+b)x + C(s-b)x = 2C(s)C(b)x = 0,$$

thus on X_0 $C(s+2b) = -C(s)$ for every $s \in R$. Integrating twice by parts, we get for $x \in X_A$

$$U(z)Ax = z^{-1}(1+e^{-2bz})^{-1} \int_0^{2b} e^{-zu} C''(u)x\, du = -x + z^2 U(z)x.$$

Now if $x \in X_0$, $\{x_n\} \subset X_A$, $x_n \to x$, then $U(z)x_n \to U(z)x$ and $AU(z)x_n \to -x + z^2 U(z)x$, thus the closedness of A implies $AU(z)x = -x + z^2 U(z)x$.

Hence for $z \in \sqrt{\varrho(A)} \cap (Z \setminus H)$ and $x \in X_0$ we get $R(z^2; A)x = U(z)x$. If $x \in X_0$ is fixed, then the function $(1+e^{-2bz})zU(z)x$ is an analytical continuation on $Z \setminus H$ of $f(z) = (1+e^{-2bz})zR(z^2; A)x$, consequently $h(z) = \int_0^{2b} e^{-zu} C(u)x\, du$ is the analytical continuation on Z of $f(z)$. Moreover,

$$\|h(z)\| \leqq e^{2b|\mathrm{Re}\, z|} \int_0^{2b} \|C(u)x\|\, du = Me^{2b|\mathrm{Re}\, z|},$$

which was to be proved.

On the other hand, suppose that $x \in X$ and $f(z) = (1+e^{-2bz})zR(z^2; A)x$ $\{z \in \sqrt{\varrho(A)}\}$ can be continued to the function $h(z)$ with the stated properties. Since for some $w \geqq 0$ $\mathrm{Re}\, z > w$ implies $zR(z^2; A)x = \int_0^\infty e^{-zu} C(u)x\, du$, therefore with the notation $C_2(u)x = \int_0^u \int_0^v C(t)x\, dt\, dv$ we get for $r > w$, $u > 0$ (see [2], (6.3.9))

$$C_2(u)x = (2\pi i)^{-1} \int_{r-i\infty}^{r+i\infty} e^{zu} h(z)(1+e^{-2bz})^{-1} z^{-2}\, dz,$$

for the last integral converges absolutely, by the properties of $h(z)$. Calculating residues, it can be shown that with the notation $h^* = \frac{d}{dz}[h(z)\,(1+e^{-2bz})^{-1}]_{z=0}$ we get for $u > 2b$

$$C_2(u)x = h^* + u\,\frac{h(0)}{2} + \sum_{k=-\infty}^{\infty} e^{z} k^{u} h(z_k)(2bz_k^2)^{-1}.$$

Hence $C_2(u+2b)x + C_2(u)x = 2h^* + (u+b)h(0)$ and, differentiating twice, we obtain $C(u+2b)x + C(u)x = 0$. By (1), it follows $C(u+b)C(b)x = 0$ $(u > 2b)$ and, again by (1), with $v > 3b$ we get

$$C(b)x = 2C(v)^2\,C(b)x - C(2v)\,C(b)x = 0,$$

and the proof is complete.

References

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