Inversion of block Toeplitz and Hankel matrices

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1. *Introduction.* W. TRENCH [1] derived an algorithm for inverting a hermitian Toeplitz matrix T_n with $O(n^2)$ operations rather than $O(n^3)$ as required by standard matrix inversion methods, and stated a similar algorithm for the nonhermitian case. ZOHAR [2] derived the extended algorithm in detail, and AKAIKE [3] generalized Zohar's derivation to block Toeplitz matrices.

TRENCH [4] modified his algorithm for the case when T_n is a Toeplitz band matrix, by which we mean that it has the form

$$\begin{bmatrix} a_0 & a_{-1} \dots a_{-p} & 0 & & \\ a_1 & a_0 & a_{-p} & 0 & \\ \vdots & & & \ddots & \\ a_q \ddots & q_{q-1} & & & \\ 0 \ddots & a_q & & & a_{-p} \\ & 0 & \ddots & & \\ & & a_q & a_q & a_0 \end{bmatrix}.$$

The simplification yields a recursive method for computing the first row and column of the inverse of this nth order Toeplitz band matrix with $O(n)$ operations.

Our purpose is to generalize Trench's algorithm for block matrices. Since the multiplication of the block-elements in general is not a commutative operation, we could not apply Trench's method without any change.

2. *Notations and definitions* Greek small letters denote square matrices of fixed ($l\times l$, say) types, ι denotes the identity and σ the zero matrix among them. Let

$$T_n = \begin{bmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} \cdots \varphi_{-n} \\ \varphi_1 & \varphi_0 & \varphi_{-1} \cdots \varphi_{-n+1} \\ & & \\ \varphi_n & \varphi_{n-1} & \varphi_{n-2} \cdots \varphi_0 \end{bmatrix} \tag{2.1}$$

denote the Toeplitz matrix which we shall investigate. Let

$$H_n = \begin{bmatrix} \alpha_0 & \alpha_1 & \dots \alpha_n \\ \alpha_1 & \alpha_2 & \cdots \alpha_{n+1} \\ \vdots & & \\ \alpha_n & \alpha_{n+1} & \dots \alpha_{2n} \end{bmatrix} \tag{2.2}$$

be a Hankel matrix.

We define the block-transpose, the block-symmetry and block-persymmetry for matrices and especially for vectorials.

The block-transpose B^t of a block matrix

$$B = [\beta_{j,k}]$$

is defined as

$$B^t = [\beta_{k,j}].$$

A block-matrix B is block-symmetric, if $B=B^t$. Let J denote the matrix

$$J = \begin{bmatrix} \sigma & \sigma \dots \sigma & \iota \\ \sigma & \sigma \dots \iota & \sigma \\ \vdots & \iddots & \\ \sigma & \iota \dots & \sigma \\ \iota & \sigma \dots & \sigma \end{bmatrix}.$$

It is obvious that for a matrix

$$B = \begin{bmatrix} \beta_{0,0} & \beta_{0,1} \cdots \beta_{0,n} \\ \beta_{1,0} & \beta_{1,1} \cdots \beta_{1,n} \\ \vdots & \\ \beta_{n,1} & \beta_{n,2} \cdots \beta_{n,n} \end{bmatrix}$$

we have

$$BJ = \begin{bmatrix} \beta_{0,n} & \beta_{0,n-1} \cdots \beta_{0,0} \\ \beta_{1,n} & \beta_{1,n-1} \quad \beta_{1,0} \\ \vdots & \\ \beta_{n,n} & \beta_{n,n-1} \quad \beta_{n,0} \end{bmatrix}$$

and

$$JB = \begin{bmatrix} \beta_{n,0} & \beta_{n,1} & \cdots \beta_{n,n} \\ \beta_{n-1,0} & \beta_{n-1,1} & \cdots \beta_{n-1,n} \\ \vdots & \vdots & \vdots \\ \beta_{0,0} & \beta_{0,1} & \cdots \beta_{0,n} \end{bmatrix}$$

i.e. in BJ the rows of B and in JB the columns of B are reversed. Consequently

$$JBJ = \begin{bmatrix} \beta_{n,n} & \beta_{n,n-1} & \cdots \beta_{n,0} \\ \beta_{n-1,n} & \beta_{n-1,n-1} & \cdots \beta_{n-1,0} \\ & & \\ \beta_{n,0} & \beta_{n-1,0} & \dots \beta_{0,0} \end{bmatrix}.$$

Definition. B is block-persymmetric if $JBJ=B^t$.

Note that all block-Toeplitz-matrices are persymmetric. If H_n is a block Hankel matrix — see (2.2) — then

$$H_n J = \begin{bmatrix} \alpha_n & \alpha_{n-1} & \cdots \alpha_0 \\ \alpha_{n+1} & \alpha_n & \cdots \alpha_1 \\ \vdots & \vdots & \vdots \\ \alpha_{2n} & \alpha_{2n-1} & \cdots \alpha_n \end{bmatrix}$$

is a block-Toeplitz matrix.

We observe that

$$J = J^t = J^{-1}.$$

Furthermore, it is clear that

$$J\begin{bmatrix}\gamma_0\\ \vdots\\ \gamma_n\end{bmatrix} = \begin{bmatrix}\gamma_n\\ \vdots\\ \gamma_0\end{bmatrix}$$

and

$$[\gamma_0, \dots, \gamma_n]J = [\gamma_n, \dots, \gamma_0].$$

3. *Recursion formulas for two vectorials*

Let

$$T_m = [\varphi_{r-s}]_{r,s=0}^{m} \tag{3.1}$$

be defined for $m=0, 1, \dots, n$. They are principal block minors of T_n.

We assume that T_m are regular.

For the sake of brevity we use the notation;

$$f_{m,+}^t = [\varphi_1, \dots, \varphi_{m+1}]; \quad f_{m,-}^t = [\varphi_{-(m+1)}, \dots, \varphi_{-1}], \tag{3.2}$$

$$s_m = T_m^{-1} f_{m,+}; \quad r_m = T_m^{+1} f_{m,-}. \tag{3.3}$$

Let the block-elements of s_m and r_m denote by $\psi_{v,m}$, $\varrho_{v,m}$ respectively, i.e.

$$s_m^t = [\psi_{0,m}, \dots, \psi_{m,m}]; \quad r_m^t = [\varrho_{0,m}, \dots, \varrho_{m,m}]. \tag{3.4}$$

We shall give a recursion formula for the computation of s_m and r_m. Suppose that s_m and r_m are known.

We can write T_{m+1} in two hyper-block forms;

$$T_{m+1} = \left[\begin{array}{c|c} T_m & f_{m,-} \\ \hline f_{m,+}^t J & \varphi_0 \end{array}\right] = \tag{3.5}$$

$$= \left[\begin{array}{c|c} \varphi_0 & f_{m,-}^t J \\ \hline f_{m,+} & T_m \end{array}\right]. \tag{3.6}$$

From (3.3) we get

$$T_{m+1} s_{m+1} = f_{m+1,+}.$$

Hence, by (3.5) we get

$$T_m \begin{bmatrix}\psi_{0,m+1}\\ \vdots\\ \psi_{m,m+1}\end{bmatrix} + f_{m,-} \cdot \psi_{m+1,m+1} = f_{m,+}, \tag{3.7}$$

$$(f_{m,+}^t J)\begin{bmatrix}\psi_{0,m+1}\\ \vdots\\ \psi_{m,m+1}\end{bmatrix} + \varphi_0 \cdot \psi_{m+1,m+1} = \varphi_{m+2}. \tag{3.8}$$

Multiplying by T_m^{-1} in (3.7), we get

$$\begin{bmatrix}\psi_{0,m+1}\\ \vdots \\ \psi_{m,m+1}\end{bmatrix} = s_m - r_m \cdot \psi_{m+1,m+1}. \tag{3.9}$$

We substitute this into (3.8) and get

$$(\varphi_0 - (f_{m,+}^t J) r_m)\psi_{m+1,m+1} = \varphi_{m+2} - (f_{m,+}^t J) s_m. \tag{3.10}$$

From the regularity of T_{m+1} it follows that the equation $T_{m+1}s_{m+1} = f_{m+1,+}$ has a unique solution, therefore the solution of (3.10) (concerning $\psi_{m+1,m+1}$) is unique. Hence the matrix

$$\varkappa_{m+1} = \varphi_0 - (f_{m,+}^t J) r_m \tag{3.11}$$

is invertible and

$$\psi_{m+1,m+1} = \varkappa_{m+1}^{-1}\left(\varphi_{m+2} - (f_{m,+}^t J)\begin{bmatrix}\psi_{0,m}\\ \vdots \\ \psi_{m,m}\end{bmatrix}\right). \tag{3.12}$$

(3.9) and (3.12) are an explicite algorithm for the compution of s_{m+1}.

For the computation of r_{m+1} we use (3.6). Since $T_{m-1} \cdot r_{m+1} = f_{m+1,-}$, we have

$$\varphi_0 \varrho_{0,m+1} + (f_{m,-}^t J)\begin{bmatrix}\varrho_{1,m+1}\\ \vdots \\ \varrho_{m+1,m+1}\end{bmatrix} = \varphi_{-m+2}, \tag{3.13}$$

$$f_{m,+}\varrho_{0,m+1} + T_m \begin{bmatrix}\varrho_{1,m+1}\\ \vdots \\ \varrho_{m+1,m+1}\end{bmatrix} = f_{m,-}. \tag{3.14}$$

From the last equation, multiplying it by T_m^{-1}, we deduce

$$\begin{bmatrix}\varrho_{1,m+1}\\ \vdots \\ \varrho_{m+1,m+1}\end{bmatrix} = r_m - s_m \cdot \varrho_{0,m+1}. \tag{3.15}$$

Substituting this into (3.13), we get easily that

$$\varrho_{0,m+1} = \lambda_{m+1}^{-1}(\varphi_{-(m+2)} - (f_{m,-}^t J) r_m), \tag{3.16}$$

where

$$\lambda_{m+1} = \varphi_0 - (f_m^t, J) s_m. \tag{3.17}$$

4. *The same vectorials for the transpose of T_m*

For a general matrix A let A^T denote the transpose of it. It is obvious that

$$T_m^T = \begin{bmatrix} \varphi_0^T & \varphi_1^T & \dots \varphi_m^T \\ \varphi_{-1}^T & \varphi_0^T & \dots \varphi_{m-1}^T \\ \vdots & \vdots & \vdots \\ \varphi_m^T & \varphi_{m-1}^T \dots & \varphi_0^T \end{bmatrix} \tag{4.1}$$

which is a block Toeplitz matrix, too.

Let $g_{m,-}$ and $g_{m,+}$ be defined by

$$g_{m,-}^t = [\varphi_{-1}^T, \dots, \varphi_{-(m+1)}^T]; \quad g_{m,+}^t = [\varphi_{m+1}^T, \dots, \varphi_1^T]. \tag{4.2}$$

Let x_m and y_m be defined from

$$T_m^T x_m = g_{m,-}; \quad T_m^T y_m = g_{m,+}. \tag{4.3}$$

Let $\Theta_{v,m}$ and $\tau_{v,m}$ denote the block elements of x_m and y_m, respectively, i.e.

$$x_m^t = [\Theta_{0,m}, \dots, \Theta_{m,m}]; \quad y_m^t = [\tau_{0,m}, \dots, \tau_{m,m}]. \tag{4.4}$$

To give recursion formulas for x_m and y_m we need to write into the formulas of § 3 φ_{-j}^T instead of φ_j, $\Theta_{j,m}$ instead of $\psi_{j,m}$, $\tau_{j,m}$ instead of $\varrho_{j,m}$ and we get

$$\begin{bmatrix} \Theta_{0,m+1} \\ \vdots \\ \Theta_{m,m+1} \end{bmatrix} = x_m - y_m \cdot \Theta_{m+1,m+1}, \tag{4.5}$$

where

$$\begin{aligned} \Theta_{m+1,m+1} &= \varepsilon_{m+1}^{-1}(\varphi_{-(m+2)}^T - (g_{m,-}^t J) x_m), \\ \varepsilon_{m+1} &= \varphi_0^T - (g_m^t \, _- J) y_m. \end{aligned} \tag{4.6}$$

Similarly we have

$$\begin{bmatrix} \tau_{1,m+1} \\ \vdots \\ \tau_{m+1,m+1} \end{bmatrix} = y_m - x_m \cdot \tau_{0,m+1}, \tag{4.7}$$

$$\begin{aligned} \tau_{0,m+1} &= \delta_{m+1}^{-1}(\varphi_{m+2}^T - (g_{m,+}^t J) y_m), \\ \delta_{m+1} &= \varphi_0^T - (g_{m,+}^t J) x_m. \end{aligned} \tag{4.8}$$

5. *Computation of the inverse of T_n*

Let

$$T_n^{-1} = [\beta_{rs}]_{r,s=0}^n, \tag{5.1}$$

and

$$(T_n^T)^{-1} = [\gamma_{rs}]_{r,s=0}^n. \tag{5.2}$$

First of all we observe that

$$\beta_{sr}^T = \gamma_{rs} \quad (r, s = 0, 1, \ldots, n). \tag{5.3}$$

First we calculate the first column and row of T_n^{-1}.

$$\left[\begin{array}{c|c} \varphi_0 & \varphi_{-1} \cdots \varphi_{-n} \\ \hline \varphi_1 & \\ \vdots & T_{n-1} \\ \varphi_n & \end{array}\right] \begin{bmatrix} \beta_{00} \\ \hline \beta_{10} \\ \\ \beta_{n0} \end{bmatrix} = \begin{bmatrix} \iota \\ \hline \sigma \\ \vdots \\ \sigma \end{bmatrix}. \tag{5.4}$$

Hence

$$\varphi_0 \cdot \beta_{00} + [\varphi_{-1}, \ldots, \varphi_{-n}] \begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = \iota \tag{5.5}$$

$$\begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{bmatrix} \beta_{00} + T_{n-1} \begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \end{bmatrix}. \tag{5.6}$$

Observing that T_{n-1} is non-singular, and that $T_{n-1} s_{n-1} = f_{n-1,+}$, from the last equation — multiplying it by T_{n-1}^{-1} — we get

$$\begin{bmatrix} \beta_{10} \\ \vdots \\ \beta_{n0} \end{bmatrix} = -s_{n-1} \beta_{0,0}. \tag{5.7}$$

Substituting it into (5.5) we deduce that

$$\begin{gathered} \beta_{00} = \alpha^{-1}, \\ \alpha = \varphi_0 - (f_{n-1,-}^t J) s_{n-1}. \end{gathered} \tag{5.8}$$

In the same way we get

$$\left[\begin{array}{c|c} \varphi_0^T & \varphi_1^T \ldots \varphi_n^T \\ \hline \varphi_{-1}^T & \\ \vdots & T_{n-1}^T \\ \varphi_{-n}^T & \end{array}\right] \begin{bmatrix} \gamma_{00} \\ \hline \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = \begin{bmatrix} \iota \\ \hline \sigma \\ \vdots \\ \sigma \end{bmatrix}. \tag{5.9}$$

Hence

$$\begin{bmatrix} \varphi_{-1}^T \\ \vdots \\ \varphi_{-n}^T \end{bmatrix} \gamma_{00} + T_{n-1}^T \begin{bmatrix} \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \end{bmatrix},$$

and multiplying by $(T_{n-1}^T)^{-1}$ we have

$$\begin{bmatrix} \gamma_{10} \\ \vdots \\ \gamma_{n0} \end{bmatrix} = -x_{n-1} \cdot \gamma_{00}. \tag{5.10}$$

We need not calculate γ_{00}, since $\gamma_{00}=\beta_{00}^T$. From (5.10) and (5.3) we get

$$\beta_{0r} = -\beta_{00}\cdot\Theta_{r-1,n-1}^T \quad (r=1,2,\dots,n). \tag{5.11}$$

Now we consider two consecutive columns in T_n^{-1}. Let k_s denote the block vector the $(s+1)$'th block of which is ι and all of the others are zero, i.e.

$$k_s = \begin{bmatrix} \sigma \\ \vdots \\ \sigma \\ \iota \\ \sigma \\ \vdots \\ \sigma \end{bmatrix} \begin{matrix} \to \text{0-th place} \\ \\ \\ \to (s+1)\text{-th place.} \\ \\ \\ \\ \end{matrix}$$

If we take

$$k_s = \left[\frac{\sigma}{l}\right],$$

then

$$k_{s-1} = \left[\frac{l}{\sigma}\right].$$

We have

$$\left[\begin{array}{c|c} T_{n-1} & \begin{matrix}\varphi_{-n}\\ \vdots \\ \varphi_{-1}\end{matrix} \\ \hline \varphi_n \cdots \varphi_1 & \varphi_0 \end{array}\right] \left[\begin{array}{c} \beta_{0,s-1} \\ \vdots \\ \beta_{n-1,s-1} \\ \hline \beta_{n,s-1} \end{array}\right] = \left[\begin{array}{c} l \\ \hline 0 \end{array}\right] \tag{5.12}$$

and that

$$\left[\begin{array}{c|c} \varphi_0 & \varphi_{-1}\cdots\varphi_{-n} \\ \hline \begin{matrix}\varphi_1\\ \vdots \\ \varphi_n\end{matrix} & T_{n-1} \end{array}\right] \left[\begin{array}{c} \beta_{0s} \\ \hline \beta_{1s} \\ \vdots \\ \beta_{ns} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline l \end{array}\right]. \tag{5.13}$$

Therefore, from (5.12) and (5.13) we get respectively, that

$$T_{n-1}\begin{bmatrix}\beta_{0,s-1}\\ \vdots \\ \beta_{n-1,s-1}\end{bmatrix} + \begin{bmatrix}\varphi_{-n}\\ \\ \varphi_{-1}\end{bmatrix}\beta_{n,s-1} = l,$$

$$\begin{bmatrix}\varphi_1\\ \vdots \\ \varphi_n\end{bmatrix}\beta_{0s} + T_{n-1}\begin{bmatrix}\beta_{1s}\\ \vdots \\ \beta_{ns}\end{bmatrix} = l.$$

Subtracting these equations

$$T_{n-1}\begin{bmatrix}\beta_{1s}-\beta_{0,s-1}\\ \vdots \\ \beta_{ns}-\beta_{n-1,s-1}\end{bmatrix} = \begin{bmatrix}\varphi_{-n}\\ \vdots \\ -1\end{bmatrix}\beta_{n,s-1} - \begin{bmatrix}\varphi_1\\ \vdots \\ \varphi_n\end{bmatrix}\beta_{0s},$$

and so, multiplying the equation by T_{n-1}^{-1}, we get

$$\begin{bmatrix}\beta_{1s}\\ \vdots \\ \beta_{ns}\end{bmatrix} = \begin{bmatrix}\beta_{0,s-1}\\ \vdots \\ \beta_{n-1,s-1}\end{bmatrix} + r_{n-1}\cdot\beta_{n,s-1} - s_n\cdot\beta_{0s}. \tag{5.14}$$

Using this formula for $s=1, \dots, n$ we get all of the elements of T_n^{-1}.

6. *Modification of the algorithm for band matrices.*

Suppose that $\varphi_r=\sigma$ when $r<-q$ or $r>p$, and that φ_{-q} and φ_p are regular matrices. In this case we can give a simplified algorithm for the computation of s_m, r_m, y_m, x_m. We shall consider only the computation of s_m and r_m.

Assume that $p+q\leqq m$. Let $s=\max(p,q)-1$. Suppose that $\psi_{0,m}, \dots, \psi_{s,m}$; $\psi_{m-s+1,m}, \dots, \psi_{m,m}$; $\varrho_{0,m}, \dots, \varrho_{s,m}$; $\varrho_{m-s+1,m}, \dots, \varrho_{m,m}$ have been computed. From the relations (3.9), (3.11), (3.12), (3.15), (3.16), (3.17) we have immadiately

$$\begin{bmatrix}\psi_{0,m+1}\\ \vdots \\ \psi_{s,m+1}\end{bmatrix} = \begin{bmatrix}\psi_{0,m}\\ \vdots \\ \psi_{s,m}\end{bmatrix} - \begin{bmatrix}\varrho_{0,m}\\ \vdots \\ \varrho_{s,m}\end{bmatrix}\psi_{m+1,m+1}$$

$$\begin{bmatrix}\varrho_{1,m+1}\\ \\ \varrho_{s,m+1}\end{bmatrix} = \begin{bmatrix}\varrho_{0,m}\\ \\ \varrho_{s-1,m}\end{bmatrix} - \begin{bmatrix}\psi_{0,m}\\ \\ \psi_{s-1,m}\end{bmatrix}\varrho_{0,m+1},$$

$$\varkappa_{m+1} = \varphi_0 - \sum_{j=1}^{p}\varphi_j\varrho_{m-j+1,m}$$

$$\lambda_{m+1} = \varphi_0 - \sum_{j=1}^{q}\varphi_{-j}\psi_{j-1,m}$$

$$\psi_{m+1,m+1} = \varkappa_{m+1}^{-1}\left(-\sum_{j=1}^{p}\varphi_j\psi_{m+1-j,m}\right),$$

$$\varrho_{0,m+1} = \lambda_{m+1}^{-1}\left(-\sum_{j=1}^{q}\varphi_{-j}\varrho_{j-1,m}\right),$$

$$\begin{bmatrix}\psi_{m-s+1,m+1}\\ \vdots \\ \psi_{m,m+1}\end{bmatrix} = \begin{bmatrix}\psi_{m-s+1,m}\\ \vdots \\ \psi_{m,m}\end{bmatrix} - \begin{bmatrix}\varrho_{m-s+1,m}\\ \vdots \\ \varrho_{m,m}\end{bmatrix}\psi_{m+1,m+1}$$

$$\begin{bmatrix}\varrho_{m+1-s,m+1}\\ \vdots \\ \varrho_{m+1,m+1}\end{bmatrix} = \begin{bmatrix}\varrho_{m-s,m}\\ \vdots \\ \varrho_{m,m}\end{bmatrix} - \begin{bmatrix}\psi_{m-s,m}\\ \vdots \\ \psi_{m,m}\end{bmatrix}\varrho_{0,m+1}.$$

For the corresponding components of x_m, y_m we can deduce similar formulas.

Now we give a simple recursion formula for the computation of the other components of $s_{n-1}, r_{n-1}, x_{n-1}, y_{n-1}$. For this we take

$$T_{n-1} = \left[\begin{array}{ccc} \varphi_0 & \cdots \varphi_{-q} & \\ \vdots & \ddots & \\ \varphi_{p-1} & & \ddots \\ \hline \varphi_p & & \ddots \\ & \ddots & \\ & \ddots & \varphi_{-q} \\ \hline & \ddots & \varphi_{q+1} \\ & \ddots & \vdots \\ & \varphi_p & \cdots \varphi_0 \end{array}\right] \begin{array}{l} \updownarrow p \\ \\ \\ \updownarrow q \end{array}$$

and corresponding to them

$$s_{n-1}^t = (\psi_{0,n-1}, \dots, \psi_{p-1,n-1} | \psi_{p,n-1}, \dots, \psi_{n-1-q,n-1} | \psi_{n-q,n-1}, \dots, \psi_{n-1})$$

$$f_{n-1,+}^t = (\varphi_1, \dots, \varphi_p | \sigma, \dots, \sigma | \sigma, \dots, \sigma).$$

From $T_{n-1} s_{n-1} = f_{n-1,+}$ we get immediately that

$$\sum_{i=-q}^{p} \varphi_i \psi_{r-i,n-1} = \sigma \quad (r = p, p+1, \dots, n-1-q).$$

The same way we have

$$\sum_{i=-q}^{p} \varphi_i \varrho_{r-1,n-1} = \sigma \quad (r = p, p+1, \dots, n-1-q).$$

Hence we have

$$\psi_{t,n-1} = -\varphi_q^{-1} \sum_{j=-p}^{q-1} \varphi_{-j} \psi_{t-q+j,n-1}$$

$$\varrho_{t,n-1} = -\varphi_{-q}^{-1} \sum_{j=-p}^{q-1} \varphi_{-j} \varrho_{t-q+j,n-1}$$

for $t \geqq s+1$.

References

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