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On the means of an entire function of several complex variables represented by multiple Dirichlet series

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1. Let

$$f(s_1,s_2)=\sum_{m,n=1}^{\infty} a_{m,n}\ \exp(s_1\lambda_m+s_2\mu_n) \tag{1.1}$$

$(s_j=\sigma_j+it_j,\quad j=1,2)$, where $a_{m,n}\in C$, the field of complex numbers, and λ_m's, μ_n's are real; $0<\lambda_1<\lambda_2<\ldots<\lambda_m\to\infty$; $0<\mu_1<\mu_2<\ldots<\mu_n\to\infty$.

It has been proved [1] that if

$$\lim_{m\to\infty}\frac{\log m}{\lambda_m}=0,\qquad \lim_{n\to\infty}\frac{\log n}{\mu_n}=0, \tag{1.2}$$

then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence. Also, Sarkar [2, pp.99] has shown that the necessary and sufficient condition for the series (1.1) satisfying (1.2) to be entire is that

$$\lim_{(m,n)\to\infty}\frac{\log|a_{m,n}|}{\lambda_m+\mu_n}=-\infty. \tag{1.3}$$

Let the family of all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3) be denoted by F. Then $f\in F$ denotes an entire function over C^2. The results can be extented to several complex variables.

Corresponding to an $f\in F$, the maximum modulus $M=M_f$ and the maximum term $\mu=\mu_f$ on R^2 are defined [2, pp.100] as

$$M(\sigma)=M_f(\sigma_1,\sigma_2)=\max\left\{|f(s_1,s_2)|,\ s_1,s_2\in C,\ \mathrm{Re}s_1=\sigma_1,\ \mathrm{Re}s_2=\sigma_2\right\}$$

$$\mu(\sigma)=\mu_f(\sigma_1,\sigma_2)=\max_{(m,n)\in N^2}\left\{|a_{m,n}|\exp(\sigma_1\lambda_m+\sigma_2\mu_n)\right\},$$

where N is the set of natural numbers. The following two lemmas are due to Sarkar [2, pp.101].

Lemma A. *Let $f \in F$ be of finite order. Then $\rho = (\rho_1, \rho_2) \gg (0,0)$ is an order point of f iff*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \operatorname{Sup}\left[\frac{\log\log M(\sigma)}{\log(e^{\sigma_1\rho_1}+e^{\sigma_2\rho_2})}\right] = 1, \qquad \sigma \in R^2.$$

Lemma B. *Let $f \in F$ be of finite order. Then $\tau = (\tau_1, \tau_2) \gg (0,0)$ is a type point of f iff*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \operatorname{Sup}\left[\frac{\log M(\sigma)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right] = 1, \quad \sigma \in R^2.$$

We define

$$I_p(\sigma_1,\sigma_2) = \lim_{(T_1,T_2)\to\infty} \frac{1}{4T_1T_2} \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} |f(\sigma_1+it_1,\sigma_2+it_2)|^p dt_1 dt_2 \tag{1.4}$$

$$m_{p,q}(\sigma_1,\sigma_2) = \frac{4}{e^{q\sigma_1}e^{q\sigma_2}} \int_0^{\sigma_1}\int_0^{\sigma_2} I_p(x_1,x_2) e^{qx_1} e^{qx_2} dx_1 dx_2, \tag{1.5}$$

where p and q are any positive numbers. We shall study some growth and asymptotic properties of $I_p(\sigma_1,\sigma_2)$ and $m_{p,q}(\sigma_1,\sigma_2)$.

2. Theorem 1. *If $f(s_1,s_2)$, $f \in F$ is of finite order, then*

$$\log M(\sigma_1,\sigma_2) \sim \log \mu(\sigma_1,\sigma_2). \tag{2.1}$$

Proof. From the definition of $M(\sigma_1,\sigma_2)$, we have

$$M(\sigma_1,\sigma_2) \le \sum_{m,n=1}^{\infty} |a_{m,n}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$$

$$= \left[\sum_{m=1}^{m_0}\sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty}\sum_{n=1}^{n_0} + \sum_{m=1}^{m_0}\sum_{n=n_0+1}^{\infty} + \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty}\right] \times$$

$$\times |a_{m,n}| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$$

$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4, \qquad \text{(say)} \tag{2.2}$$

Now

$$\sum\nolimits_1 \le A\ \exp(\sigma_1\lambda_{m_0} + \sigma_2\mu_{n_0})$$

$$\begin{aligned}\sum\nolimits_2 &= \sum_{m=m_0+1}^{\infty}\sum_{n=1}^{n_0}|a_{m,n}|\exp(\sigma_1\lambda_m + \sigma_2\mu_n)\\ &< B\exp(\sigma_2\mu_{n_0})\sum_{m=m_0+1}^{\infty}|a_{m,n}|\exp(\sigma_1\lambda_m)\\ &\quad= B\exp(\sigma_2\mu_{n_0})\sum_{m=m_0+1}^{\infty}|a_{m,n}|\exp(\sigma_1+2\varepsilon)\lambda_m\exp(-2\varepsilon\lambda_m)\,.\end{aligned}$$

Similarly

$$\sum\nolimits_3 < C\exp(\sigma_1\lambda_{m_0})\sum_{n=n_0+1}^{\infty}|a_{m,n}|\exp(\sigma_2+2\varepsilon)\exp(-2\varepsilon\mu_n)\,.$$

Also

$$\begin{aligned}\sum\nolimits_4 &= \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty}|a_{m,n}|\exp(\sigma_1\lambda_m + \sigma_2\mu_n)\\ &= \sum_{m>m_0}\sum_{n>n_0}|a_{m,n}|\exp\{(\sigma_1+2\varepsilon)\lambda_m\}\exp\{(\sigma_2+2\varepsilon)\mu_n\}\\ &\quad\times\exp(-2\varepsilon\lambda_m)\exp(-2\varepsilon\mu_n)\,.\end{aligned}$$

Using (1.2), we have for a given $\varepsilon > 0$,

$$\sum\nolimits_2 < B\exp(\sigma_2\mu_{n_0})\sum_{m=m_0+1}^{\infty}|a_{m,n}|\exp\{(\sigma_1+2\varepsilon)\lambda_m\}\frac{1}{m^2}$$

$$\sum\nolimits_3 < C\exp(\sigma_1\lambda_{m_0})\sum_{m=n_0+1}^{\infty}|a_{m,n}|\exp\{(\sigma_2+2\varepsilon)\mu_n\}\frac{1}{n^2}$$

and

$$\begin{aligned}\sum\nolimits_4 &< \sum_{m>m_0}\sum_{n>n_0}|a_{m,n}|\exp\{(\sigma_1+2\varepsilon)\lambda_m\}\exp\{(\sigma_2+2\varepsilon)\mu_n\}\frac{1}{m^2n^2}\\ &< K_1\mu(\sigma_1+2\varepsilon,\sigma_2+2\varepsilon)\sum_{m>m_0}\sum_{n>n_0}\frac{1}{m^2n^2}\,.\end{aligned}$$

Hence (2.2) gives

$$M(\sigma_1,\sigma_2) \le A\exp(\sigma_1\lambda_{m_0}+\sigma_2\mu_{n_0}) \tag{2.3}$$
$$+ B\exp(\sigma_2\mu_{n_0}) \sum_{m=m_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_1+2\varepsilon)\lambda m\}\frac{1}{m^2}$$
$$+ C\exp(\sigma_1\lambda_{m_0}) \sum_{n=n_0+1}^{\infty} |a_{m,n}| \exp\{(\sigma_2+2\varepsilon)\mu_n\}\frac{1}{n^2}$$
$$+ K_1\mu(\sigma_1+2\varepsilon,\sigma_2+2\varepsilon) \sum_{m>m_0} \sum_{n>n_0} \frac{1}{m^2n^2}$$
$$\le 0(\mu(\sigma_1+2\varepsilon,\sigma_2+2\varepsilon)) \,. \tag{2.4}$$

We also note that $\mu(\sigma_1,\sigma_2) \le M(\sigma_1,\sigma_2)$. Hence $\log M(\sigma_1,\sigma_2) \sim \log\mu(\sigma_1,\sigma_2)$.

3. Theorem 2. *If $f(s_1,s_2)$, $f\in F$ is of finite order, then for $p\ge 1$,*

$$\log I_p(\sigma_1,\sigma_2) \sim p\log M(\sigma_1,\sigma_2). \tag{3.1}$$

We first prove the following

Lemma 1. *For $f(s_1,s_2)$, $f\in F$*

$$a_{m,n}\exp(\sigma_1\lambda_m+\sigma_2\mu_n) = \lim_{(T_1,T_2)\to\infty} \frac{1}{4T_1T_2} \times \tag{3.2}$$
$$\times \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} [\exp(-i(t_1\lambda_m+t_2\mu_n)) \times f(\sigma_1+it_1,\sigma_2+it_2)]dt_1dt_2 \,.$$

Proof. We have

$$f(s_1,s_2) = f(\sigma_1+it_1,\sigma_2+it_2)$$
$$= \sum_{M,N=1}^{\infty} a_{M,N}\exp\{(\sigma_1+it_1)\lambda_M+(\sigma_2+it_2)\mu_N\}$$
$$= a_{m,n}\exp\{(\sigma_1+it_1)\lambda_m+(\sigma_2+it_2)\mu_n\}$$
$$+ \sum_{M\ne m}\sum_{N\ne n} a_{M,N}\exp\{(\sigma_1+it_1)\lambda_M+(\sigma_2+it_2)\mu_N\}$$

Also

$$\exp\{-i(t_1\lambda_m+t_2\mu_n)\}f(\sigma_1+it_1,\sigma_2+it_2) =$$

$$= a_{m,n}\exp(\sigma_1\lambda_m+\sigma_2\mu_n)+\sum_{M\neq m}\sum_{N\neq n}a_{M,N}\exp\{(\sigma_1\lambda_M+\sigma_2\mu_N)\}\times$$
$$\times\exp\{it_1(\lambda_M-\lambda_m)+it_2(\mu_N-\mu_n)\}\,.$$

Since the series on the right is uniformly convergent for any finite t_1 and t_2 ranges, therefore we may integrate term by term for finite t_1 and t_2. Hence on integration, all the terms, for which $m\neq M$, $n\neq N$, after dividing by T_1T_2 vanish as $T_1,\ T_2\to\infty$ and we obtain

$$\lim_{(T_1,T_2)\to\infty}\frac{1}{4T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\exp\{-i(t_1\lambda_m+t_2\mu_n)\}f(\sigma_1+it_1,\sigma_2+it_2)dt_1dt_2$$
$$= a_{m,n}\exp(\sigma_1\lambda_m+\sigma_2\mu_n)\,.$$

PROOF of Theorem 2. From (1.4), we have

$$I_p(\sigma_1,\sigma_2)=\lim_{(T_1,T_2)\to\infty}\frac{1}{4T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|^p dt_1dt_2$$
$$\leq\{M(\sigma_1,\sigma_2)\}^p\,. \tag{3.3}$$

Also from Lemma 1, we get

$$\mu(\sigma_1,\sigma_2)\leq\lim_{(T_1,T_2)\to\infty}\frac{1}{4T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|dt_1dt_2\,. \tag{3.4}$$

Applying Hölder's inequality for $p>1$, we have

$$\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|dt_1dt_2$$
$$\leq\int_{-T_1}^{T_1}\left[\left\{\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|^p dt_2\right\}^{1/p}\left\{\int_{-T_2}^{T_2}dt_2\right\}^{1-\frac{1}{p}}\right]dt_1$$
$$\leq\left[\int_{-T_1}^{T_1}\left\{\left(\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|^p dt_2\right)^{\frac{1}{p}}\right\}^p dt_1\right]^{\frac{1}{p}}\times$$

$$\times \left[\int_{-T_1}^{T_1} \left\{ \left(\int_{-T_2}^{T_2} dt_2 \right)^{1-\frac{1}{p}} \right\}^{\frac{p}{p-1}} dt_1 \right]^{1-\frac{1}{p}}$$

$$= \left[\int_{-T_1}^{T_1} \left\{ \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_2 \right\} dt_1 \right]^{\frac{1}{p}} \left[\left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_2 \right\} dt_1 \right]^{1-\frac{1}{p}}$$

$$= \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}} \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_1 dt_2 \right\}^{1-\frac{1}{p}} .$$

Therefore,

$$(3.5)\quad \mu(\sigma_1, \sigma_2) \le \lim_{(T_1,T_2)\to\infty} \frac{1}{4T_1T_2}$$

$$\times \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}} \left\{ \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} dt_1 dt_2 \right\}^{1-\frac{1}{p}}$$

$$(3.6)\quad = \lim_{(T_1,T_2)\to\infty} \left\{ \frac{1}{4T_1T_2} \int_{-T_1}^{T_1} \int_{-T_2}^{T_2} |f(\sigma_1 + it_1, \sigma_2 + it_2)|^p dt_1 dt_2 \right\}^{\frac{1}{p}}$$

$$= \left\{ I_p(\sigma_1, \sigma_2) \right\}^{\frac{1}{p}} .$$

For, $p = 1$, we have from (4.4)

$$(3.7)\qquad \mu(\sigma_1, \sigma_2) \quad \le \quad I_1(\sigma_1, \sigma_2)$$

Hence for $p \ge 1$, we have

$$(3.8)\qquad \mu(\sigma_1, \sigma_2) \le \left\{ I_p(\sigma_1, \sigma_2) \right\}^{\frac{1}{p}} \le M(\sigma_1, \sigma_2),$$

and using (2.1), we get the theorem.

Corollary 1. *If* $f(s_1, s_2)$, $f \in F$, *is of finite order* $\rho = (\rho_1, \rho_2) \gg (0, 0)$, *then for* $p \ge 1$

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \operatorname{Sup} \left\{ \frac{\log\log I_p(\sigma_1, \sigma_2)}{\log(e^{\sigma_1\rho_1} + e^{\sigma_2\rho_2})} \right\} = 1 \ .$$

Corollary 2. *If $f(s_1, s_2)$, $f \in F$ is of finite order $\rho = (\rho_1, \rho_2) \gg (0,0)$ and type $\tau = (\tau_1, \tau_2) \gg (0,0)$, then for $p \geq 1$*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \operatorname{Sup}\left\{ \frac{\log I_p(\sigma_1, \sigma_2)}{(\tau_1 e^{\sigma_1\rho_1} + \tau_2 e^{\sigma_2\rho_2})} \right\} = p \ .$$

4. Theorem 3. *If $f(s_1, s_2)$, $f \in F$ is of finite order $\rho = (\rho_1, \rho_2) \gg (0,0)$ then for $p \geq 1$*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \operatorname{Sup}\left\{ \frac{\log\log m_{p,q}(\sigma_1, \sigma_2)}{\log(e^{\sigma_1\rho_1} + e^{\sigma_2\rho_2})} \right\} = 1 \ . \tag{4.1}$$

Proof. We use (3.8) in (1.5) and then with the help of (2.4) we get

$$\begin{aligned} m_{p,q}(\sigma_1, \sigma_2) &= \frac{4}{e^{q\sigma_1}\, e^{q\sigma_2}} \int_0^{\sigma_1}\int_0^{\sigma_2} I_p(x_1, x_2) e^{qx_1}\, e^{qx_2} dx_1 dx_2 \qquad (4.2)\\ &\leq \frac{4}{e^{q\sigma_1}\, e^{q\sigma_2}} \int_0^{\sigma_1}\int_0^{\sigma_2} \{M(x_1, x_2)\}^p\, e^{qx_1}\, e^{qx_2} dx_1 dx_2 \\ &\leq \frac{4}{e^{q\sigma_1}\, e^{q\sigma_2}}\ \frac{\{M(\sigma_1, \sigma_2)\}^p}{q^2} (e^{q\sigma_1} - 1)(e^{q\sigma_2} - 1) \\ &= \frac{4}{q^2} \{M(\sigma_1, \sigma_2)\}^p\, (1 - e^{-q\sigma_1})(1 - e^{-q\sigma_2}) \\ &< \frac{4}{q^2} \{\mu(\sigma_1 + 2\varepsilon, \sigma_2 + 2\varepsilon)\}^p\, (1 - e^{-q\sigma_1})(1 - e^{-q\sigma_2}) \qquad (4.3) \end{aligned}$$

Also, using (3.8) in (1.5), we have for $h > 0$

$$\begin{aligned} &m_{p,q}(\sigma_1 + \frac{h}{\rho_1}, \sigma_2 + \frac{h}{\rho_2}) \qquad (4.4)\\ &= \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} I_p(x_1, x_2) e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\geq \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_0^{\sigma_1 + \frac{h}{\rho_1}} \int_0^{\sigma_2 + \frac{h}{\rho_2}} \{\mu(x_1, x_2)\}^p\, e^{qx_1} e^{qx_2} dx_1 dx_2 \\ &\geq \frac{4}{e^{q(\sigma_1 + \frac{h}{\rho_1})} e^{q(\sigma_2 + \frac{h}{\rho_2})}} \int_{\sigma_1}^{\sigma_1 + \frac{h}{\rho_1}} \int_{\sigma_2}^{\sigma_2 + \frac{h}{\rho_2}} \{\mu(x_1, x_2)\}^p\, e^{qx_1} e^{qx_2} dx_1 dx_2 \end{aligned}$$

$$\geq \frac{4\{\mu(\sigma_1,\sigma_2)\}^p}{e^{q(\sigma_1+\frac{h}{\rho_1})}e^{q(\sigma_2+\frac{h}{\rho_2})}} \times \frac{1}{q^2}\left\{(e^{q(\sigma_1+\frac{h}{\rho_1})}-e^{q\sigma_1})\right\}\times$$
$$\times\left\{e^{q(\sigma_2+\frac{h}{\rho_2})}-e^{q\sigma_2}\right\}$$
$$=\frac{4}{q^2}\{\mu(\sigma_1,\sigma_2)\}^p\,(1-e^{-q\frac{h}{\rho_1}})(1-e^{-q\frac{h}{\rho_2}})\ .$$

From (5.3) and (5.4), we obtain

$$\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left\{\frac{\log\log m_{p,q}(\sigma_1,\sigma_2)}{\log(e^{\sigma_1\rho_1}+e^{\sigma_2\rho_2})}\right\}=\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left\{\frac{\log\log\mu(\sigma_1,\sigma_2)}{\log(e^{\sigma_1\rho_1}+e^{\sigma_2\rho_2})}\right\}$$

Using (2.1) and Lemma A, we get

$$\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left\{\frac{\log\log m_{p,q}(\sigma_1,\sigma_2)}{\log(e^{\sigma_1\rho_1}+e^{\sigma_2\rho_2})}\right\}$$
$$=\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left\{\frac{\log\log M(\sigma_1,\sigma_2)}{\log(e^{\sigma_1\rho_1+\sigma_2\rho_2})}\right\}=1$$

and hence the result.

5. Theorem 4. *If $f(s_1,s_2)$, $f\in F$ is of finite order $\rho=(\rho_1,\rho_2)\gg(0,0)$ and type $\tau=(\tau_1,\tau_2)\gg(0,0)$ then for $p\geq 1$*

$$\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left\{\frac{\log m_{p,q}(\sigma_1,\sigma_2)}{(\tau_1e^{\sigma_1\rho_1}+\tau_2e^{\sigma_2\rho_2})}\right\}=p\ . \tag{5.1}$$

Proof. We have

$$m_{p,q}(\sigma_1,\sigma_2)=\frac{4}{e^{q\sigma_1}\,e^{q\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}I_p(x_1,x_2)e^{qx_1}\,e^{qx_2}dx_1dx_2$$
$$\leq\frac{4}{e^{q\sigma_1}\,e^{q\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}\{M(x_1,x_2)\}^p\,e^{qx_1}\,e^{qx_2}dx_1dx_2$$
$$\leq\frac{4}{q^2}\{M(\sigma_1,\sigma_2)\}^p(1-e^{-q\sigma_1})(1-e^{-q\sigma_2})\,. \tag{5.2}$$

Therefore

$$\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left[\frac{\log m_{p,q}(\sigma_1,\sigma_2)}{\tau_1e^{\sigma_1\rho_1}+\tau_2e^{\sigma_2\rho_2}}\right]\leq p\lim_{(\sigma_1,\sigma_2)\to\infty}\mathrm{Sup}\left[\frac{\log M(\sigma_1,\sigma_2)}{\tau_1e^{\sigma_1\rho_1}+\tau_2e^{\sigma_2\rho_2}}\right]$$

Also

$$m_{p,q}(\sigma_1+\frac{h}{\rho_1},\sigma_2+\frac{h}{\rho_2})=$$

$$= \frac{4}{e^{q(\sigma_1+\frac{h}{\rho_1})}e^{q(\sigma_2+\frac{h}{\rho_2})}} \int_0^{\sigma_1+\frac{h}{\rho_1}} \int_0^{\sigma_2+\frac{h}{\rho_2}} I_p(x_1,x_2)e^{qx_1}\, e^{qx_2} dx_1 dx_2$$

$$\geq \frac{4}{e^{q(\sigma_1+\frac{h}{\rho_1})}e^{q(\sigma_2+\frac{h}{\rho_2})}} \int_0^{\sigma_1+\frac{h}{\rho_1}} \int_0^{\sigma_2+\frac{h}{\rho_2}} \{\mu(x_1,x_2)\}^p\, e^{qx_1} e^{qx_2} dx_1 dx_2$$

$$\geq \frac{4}{q^2}\{\mu(\sigma_1,\sigma_2)\}^p (1-e^{-q\frac{h}{\rho_1}})(1-e^{-q\frac{h}{\rho_2}})\,.$$

Therefore

$$e^h \lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1+\frac{h}{\rho_1},\sigma_2+\frac{h}{\rho_2})}{\tau_1 e^{\rho_1(\sigma_1+\frac{h}{\rho_1})}+\tau_2 e^{\rho_2(\sigma_2+\frac{h}{\rho_2})}}\right] \geq p \lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log \mu(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right]$$

or $$e^h \lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right] \geq p \lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log \mu(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right].$$

Since h is arbitrary, we have

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log m_{p,q}(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right] \geq p \lim_{(\sigma_1,\sigma_2)\to\infty} \text{Sup} \left[\frac{\log \mu(\sigma_1,\sigma_2)}{\tau_1 e^{\sigma_1\rho_1}+\tau_2 e^{\sigma_2\rho_2}}\right]. \tag{5.3}$$

Also $\log M(\sigma_1,\sigma_2) \sim \log \mu(\sigma_1,\sigma_2)$ and therefore from (5.2) and (5.3), we get the theorem.

6. Theorem 5. *For $f(s_1,s_2)$, $\quad f \in F$ and $p \geq 1$,*

$$\lim_{(\sigma_1,\sigma_2)\to\infty} \left[\frac{1}{e^{q\sigma_1(1-\alpha_1)}e^{q\sigma_2(1-\alpha_2)} m_{p,q}(\sigma_1,\sigma_2) - m_{p,q}(\alpha_1\sigma_1,\alpha_2\sigma_2)}\right] = 0 \tag{6.1}$$

where α_1,α_2 $(0<\alpha_1,\alpha_2<1)$ are constants.

We first prove the following lemma:

Lemma 2. *Let $f(s_1,s_2)$, $\quad f\in F$, then for $(0<\sigma_1'<\bar\sigma_1<\sigma_1)$, and $(0<\sigma_2'<\bar\sigma_2<\sigma_2)$,*

$$\Big[\{\mu(\bar\sigma_1,\sigma_2')\}^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\bar\sigma_2}-e^{q\sigma_2'}) + \{\mu(\sigma_1',\bar\sigma_2)\}^p(e^{q\bar\sigma_1}-e^{q\sigma_1'})(e^{q\sigma_2}-e^{q\bar\sigma_2}) + \{\mu(\bar\sigma_1,\bar\sigma_2)\}^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\sigma_2}-e^{q\bar\sigma_2})\Big]$$
$$\le\left(\frac{q}{2}\right)^2\Big[e^{q\sigma_1+q\sigma_2}m_{p,q}(\sigma_1,\sigma_2)-e^{q\bar\sigma_1+q\bar\sigma_2}m_{p,q}(\bar\sigma_1,\bar\sigma_2)\Big]$$
$$\le\Big[\{M(\sigma_1,\bar\sigma_2)\}^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\bar\sigma_2}-1) + \{M(\bar\sigma_1,\sigma_2)\}^p(e^{q\bar\sigma_1}-1)(e^{q\sigma_2}-e^{q\bar\sigma_2}) + \{M(\sigma_1,\sigma_2)\}^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\sigma_2}-e^{q\bar\sigma_2})\Big],$$

where $p\ge1$ and q is any positive number.

Proof of Lemma 2. From (1.3), we have

$$e^{q\sigma_1+q\sigma_2}m_{p,q}(\sigma_1,\sigma_2)-e^{q\bar\sigma_1+q\bar\sigma_2}m_{p,q}(\bar\sigma_1,\bar\sigma_2)$$
$$=4\int_{\bar\sigma_1}^{\sigma_1}\int_0^{\bar\sigma_2}I_p(x_1,x_2)e^{qx_1+qx_2}dx_1dx_2+4\int_0^{\bar\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}I_p(x_1,x_2)e^{qx_1+qx_2}dx_1dx_2 + 4\int_{\bar\sigma_1}^{\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}I_p(x_1,x_2)e^{qx_1+qx_2}dx_1dx_2$$

Hence, using (3.8), we have for $p\ge1$

$$e^{q\sigma_1+q\sigma_2}m_{p,q}(\sigma_1,\sigma_2)-e^{q\bar\sigma_1+q\bar\sigma_2}m_{p,q}(\bar\sigma_1,\bar\sigma_2) \tag{6.2}$$
$$\le4\int_{\bar\sigma_1}^{\sigma_1}\int_0^{\bar\sigma_2}\{M(x_1,x_2)\}^p\,e^{qx_1+qx_2}\,dx_1dx_2+ + 4\int_0^{\bar\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\{M(x_1,x_2)\}^p\,e^{qx_1+qx_2}\,dx_1dx_2 + 4\int_{\bar\sigma_1}^{\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}\{M(x_1,x_2)\}^p\,e^{qx_1+qx_2}\,dx_1dx_2$$
$$\le\frac{4}{q^2}[M(\sigma_1,\bar\sigma_2)]^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\bar\sigma_2}-1)$$

$$+\frac{4}{q^2}[M(\bar\sigma_1,\sigma_2)]^p(e^{q\bar\sigma_1}-1)(e^{q\sigma_2}-e^{q\bar\sigma_2})$$
$$+\frac{4}{q^2}[M(\sigma_1,\sigma_2)]^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\sigma_2}-e^{q\bar\sigma_2})\,.$$

Also

$$(6.3)\qquad e^{q\sigma_1+q\sigma_2}m_{p,q}(\sigma_1,\sigma_2)-e^{q\bar\sigma_1+q\bar\sigma_2}m_{p,q}(\bar\sigma_1,\bar\sigma_2)$$
$$\geq 4\int_{\bar\sigma_1}^{\sigma_1}\int_{\sigma_2'}^{\bar\sigma_2}[\mu(x_1,x_2)]^p\,e^{qx_1+qx_2}\,dx_1dx_2$$
$$+\,4\int_{\sigma_1'}^{\bar\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}[\mu(x_1,x_2)]^p\,e^{qx_1+qx_2}\,dx_1dx_2$$
$$+\,4\int_{\bar\sigma_1}^{\sigma_1}\int_{\bar\sigma_2}^{\sigma_2}[\mu(x_1,x_2)]^p\,e^{qx_1+qx_2}\,dx_1dx_2$$
$$\geq\frac{4}{q^2}[\mu(\bar\sigma_1,\sigma_2')]^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\bar\sigma_2}-e^{q\sigma_2'})$$
$$+\frac{4}{q^2}[\mu(\sigma_1',\bar\sigma_2)]^p(e^{q\bar\sigma_1}-e^{q\sigma_1'})(e^{q\sigma_2}-e^{q\bar\sigma_2})$$
$$+\frac{4}{q^2}[\mu(\bar\sigma_1,\bar\sigma_2)]^p(e^{q\sigma_1}-e^{q\bar\sigma_1})(e^{q\sigma_2}-e^{q\bar\sigma_2})\,.$$

Combining (7.2) and (7.3) we obtain the lemma.

Proof of Theorem 5. If we put

$$\bar\sigma_1=\alpha_1\sigma_1,\quad \sigma_1'=\beta_1\sigma_1,\quad \bar\sigma_2=\alpha_2\sigma_2,\quad \sigma_2'=\beta_2\sigma_2,$$

where $\beta_1<\alpha_1,\quad \beta_2<\alpha_2$ in Lemma 2, we get

$$\Big[4\{\mu(\alpha_1\sigma_1,\beta_2\sigma_2)\}^p(e^{q\sigma_1}-e^{q\alpha_1\sigma_1})(e^{q\alpha_2\sigma_2}-e^{q\beta_2\sigma_2})$$
$$+\,4\{\mu(\beta_1\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\alpha_1\sigma_1}-e^{q\beta_1\sigma_1})(e^{q\sigma_2}-e^{q\alpha_2\sigma_2})$$
$$+\,4\{\mu(\alpha_1\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\sigma_1}-e^{q\alpha_1\sigma_1})(e^{q\sigma_2}-e^{q\alpha_2\sigma_2})\Big]$$
$$\leq q^2\Big[e^{q\sigma_1+q\sigma_2}m_{p,q}(\sigma_1,\sigma_2)-e^{q\alpha_1\sigma_1+q\alpha_2\sigma_2}m_{p,q}(\alpha_1\sigma_1,\alpha_2\sigma_2)\Big]$$
$$\leq\Big[4\{M(\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\sigma_1}-e^{q\alpha_1\sigma_1})(e^{q\alpha_2\sigma_2}-1)$$

$$+\,4\{M(\alpha_1\sigma_1,\sigma_2)\}^p(e^{q\alpha_1\sigma_1}-1)(e^{q\sigma_2}-e^{q\alpha_2\sigma_2})$$
$$+\,4\{M(\sigma_1,\sigma_2)\}^p(e^{q\sigma_1}-e^{q\alpha_1\sigma_1})(e^{q\sigma_2}-e^{q\alpha_2\sigma_2})\Big]$$

Dividing by $e^{q\alpha_1\sigma_1}\ e^{q\alpha_2\sigma_2}$, we obtain

$$\begin{aligned}
&\Big[4\{\mu(\alpha_1\sigma_1,\beta_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)}-1)(1-e^{q\sigma_2(\beta_2-\alpha_2)})\\
&\quad+4\{\mu(\beta_1\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\sigma_2(1-\alpha_2)}-1)(1-e^{q\sigma_1(\beta_1-\alpha_1)})\\
&\quad+4\{\mu(\alpha_1\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)}-1)(e^{q\sigma_2(1-\alpha_2)}-1)\Big]\\
&\le q^2\Big[e^{q\sigma_1(1-\alpha_1)}e^{q\sigma_2(1-\alpha_2)}m_{p,q}(\sigma_1,\sigma_2)-m_{p,q}(\alpha_1\sigma_1,\alpha_2\sigma_2)\Big]\\
&\le\Big[4\{M(\sigma_1,\alpha_2\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)}-1)(1-e^{-q\alpha_2\sigma_2})\\
&\quad+4\{M(\alpha_1\sigma_1,\sigma_2)\}^p(1-e^{-q\alpha_1\sigma_1})(e^{q\sigma_2(1-\alpha_2)}-1)\\
&\quad+4\{M(\sigma_1,\sigma_2)\}^p(e^{q\sigma_1(1-\alpha_1)}-1)(e^{q\sigma_2(1-\alpha_2)}-1)\Big]
\end{aligned}$$

Taking limits on both the sides, Theorem 5 follows from the above inequalities.

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References

[1] A. I. Janusauskas, Elementary theorem on the convergence of Double Dirichlet Series, *Dokl. Akad. Nauk SSSR* **234**, 1.

[2] P. K. Sarkar, On Order and type of an Entire function in several complex variables represented by Dirichlet series, *Bull. Cal. Math. Soc.* **74** (1982), 99-112.

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