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A characterization of the local structure of static stars

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Abstract. It is shown that the existence of a "spherical coweakly-affine static reference frame" in a spacetime gives rise to a 2 by 2 warped product metric tensor describing a static star locally.

1. Introduction

Intuitively, a "static star" refers to a gravitational field generated by a time independent nonrotating source. Throughout this paper, we only will consider the gravitational field exterior to the celestial body of a "static star" with no matter present, yet still call that gravitation a "static star". In the literature, formal descriptions of "static stars" are made by using asymptotic considerations (see [4, Ch. 9]). Yet the physical observations are made locally and hence, these observations are suited to build a "static star" model.

A local characterization of Schwarzschild and Reissner metrics is studied in [3] by using the (local) concepts of infinitesimal isotropy (or equivalently, null anisotropy) and weak-affinity. From the physical point of view, infinitesimal isotropy refers to "spherical symmetry" and weak-affinity, together with infinitesimal isotropy, refers to nonrotation. Indeed this is the case in Schwarzschild and Reissner metrics.

In the current theory of "static stars", staticity is expressed by the existence of a static reference frame on some open subset of the spacetime. In this paper, we will make it our starting point in order to reach

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back to infinitesimal isotropy. We will formulate a "static star" by only imposing conditions on the static reference frame, an important one is the coweak-affinity of the static reference frame. Then with some "sphericality" assumptions, we will obtain a 2 by 2 warped product decomposition of the metric locally. Yet to reach infinitesimal isotropy, we will make in addition the assumption of symmetry with respect to the stress-energy tensor. Then the spacetime will become the one described in [3], which has a 2 by 2 warped product decomposition locally.

2. Preliminaries

Let M be a 4-dimensional spacetime. A future directed unit timelike vector field Z_1 is called a reference frame. Let $\omega_1(\cdot) = \langle \cdot, Z_1 \rangle$ be the 1-form associated to Z_1. We call Z_1 an *irrotational* (or locally synchronizable) reference frame if $\omega_1 \wedge d\omega_1 = 0$. Also it can be shown that if Z_1 is irrotational, then locally there exist functions t and $h > 0$ such that $\omega_1 = -hdt$, i.e., $Z_1 = -g\nabla t$ locally (see [11, pag. 52–59]). A reference frame Z_1 is called *stationary* if there exists a function $f > 0$ such that $Z = fZ_1$ is a Killing vector field. A stationary reference frame Z_1 is called *static* if Z_1 is irrotational (see [11 pag. 219]). A spacetime M is called static (resp., stationary) if there exists a static (resp., stationary) reference frame on M. In fact, if M is a static spacetime with static reference frame Z_1, then M is locally a warped product $I_{g^2} \times N$, where $I \subset \mathbb{R}$, N is a Riemannian manifold, and $g > 0$ is a smooth function. Then $Z_1 = -g\nabla t$ and the Killing field $Z = fZ_1 = -g^2\nabla t = \frac{\partial}{\partial t}$ (cf. [8 pag. 360–361]). It follows that the orthogonal vector bundle $Z_1^\perp$ to Z_1 is integrable with totally geodesic leaves and $f = g$.

3. Static spacetimes

Lemma 3.1. *If Z is a timelike Killing vector field on M, then $\nabla_Z Z = f\nabla f \perp Z$, where $\langle Z, Z\rangle = -f^2$.*

Proof. First note that, by the Killing identity, $\langle \nabla_Z Z, Z\rangle = 0$ and hence $\nabla_Z Z \perp Z$. Also, for every $X \in \Gamma TM$,

$$\begin{aligned} 0 &= \langle \nabla_Z Z, X\rangle + \langle Z, \nabla_X Z\rangle = \langle \nabla_Z Z, X\rangle + \frac{1}{2}X\langle Z, Z\rangle \\ &= \langle \nabla_Z Z, X\rangle - \frac{1}{2}X(f^2) = \langle \nabla_Z Z - f\nabla f, X\rangle. \end{aligned}$$

Hence $\nabla_Z Z = f\nabla f$. □

Definition 3.2. A stationary reference frame Z_1 on M is called proper if $\nabla f \neq 0$ at each $p \in M$, where $Z = fZ_1$ $(f > 0)$ is a Killing vector field.

Remark 3.3. Note that the properness of Z_1 is also used for the global characterization of the static parts of the Schwarzschild and Reissner metrics (see [5] and [6]).

Definition 3.4. Let Z_1 be a proper stationary reference frame on M. The unit acceleration P_1 of Z is defined by $P_1 = \frac{\nabla f}{\|\nabla f\|}$. Also the canonical distributions W_1 and W_2 in TM are defined by $W_1 = \operatorname{span}\{Z_1, P_1\}$ and $W_2 = W_1^{\perp}$. Furthermore, if Z_1 is static then P_1 is called the fundamental gyroscope of Z_1.

Remark 3.5. Indeed, if Z_1 is static, then it can be easily shown that $F_{Z_1} P_1 = 0$ by using the fact that $Z_1^{\perp}$ has totally geodesic leaves, where F_{Z_1} is the Fermi–Walker connection over Z_1 (see [11, pag. 50–52]). Thus P_1, as being the unit acceleration of Z_1, can be named as the fundamental gyroscope of Z_1.

Lemma 3.6. *Let Z_1 be a proper stationary reference frame on M. Then*

(1) $[Z, \nabla f] = 0$ *and hence W_1 is integrable.*

(2) $[Z, P_1] = 0$ *and hence $Z\|\nabla f\| = 0$.*

Proof. (1) Let $X \perp Z$ be a vector field. Then

$$\begin{aligned}\langle \nabla_X \nabla f, Z\rangle &= -\langle \nabla f, \nabla_X Z\rangle = \langle \nabla_{\nabla f} Z, X\rangle \\ &= \langle \nabla_Z \nabla f, X\rangle + \langle [\nabla f, Z], X\rangle = \langle \nabla_X \nabla f, Z\rangle + \langle [\nabla f, Z], X\rangle\end{aligned}$$

and hence $[\nabla f, Z] \perp X$. Also,

$$\langle \nabla_Z \nabla f, Z\rangle = -\langle \nabla f, \nabla_Z Z\rangle = \langle \nabla_{\nabla f} Z, Z\rangle$$

and hence $[Z, \nabla f] \perp Z$. Thus $[Z, \nabla f] = 0$, and it also follows that W_1 is integrable.

(2) By (1), $[Z, P_1] \in \Gamma W_1$ and, since

$$\langle \nabla_Z P_1, Z\rangle = -\langle P_1, \nabla_Z Z\rangle = \langle \nabla_{P_1} Z, Z\rangle$$

and

$$\langle [Z, P_1], P_1\rangle = \langle \nabla_Z P_1, P_1\rangle - \langle \nabla_{P_1} Z, P_1\rangle = 0,$$

it follows that $[Z, P_1] = 0$ and hence $Z\|\nabla f\| = 0$. □

Lemma 3.7. *If Z_1 is a proper static reference frame on M, then W_2 is also integrable.*

PROOF. Recall from the preliminaries that locally $Z = fZ_1 = -g^2\nabla t$. Hence ∇t and ∇f are orthogonal to W_2. Then for any $X, Y \in \Gamma W_2$,

$$\begin{aligned}\langle \nabla t, [X, Y]\rangle &= \langle \nabla t, \nabla_X Y\rangle - \langle \nabla t, \nabla_Y X\rangle \\ &= -\langle \nabla_X \nabla t, Y\rangle + \langle \nabla_Y \nabla t, X\rangle = 0\end{aligned}$$

and similarly, $\langle \nabla f, [X, Y]\rangle = 0$. Thus it follows that $[X, Y] \in \Gamma W_2$. □

Definition 3.8. Let X be a vector field on M. The affinity tensor field of X is defined by

$$(\mathcal{L}_X \nabla)(U, V) = \mathcal{L}_X \nabla_U V - \nabla_U \mathcal{L}_X V - \nabla_{\mathcal{L}_X U} V,$$

where $\mathcal{L}$ is the Lie derivative. X is called affine if $\mathcal{L}_X \nabla = 0$ (see [10, pag. 103]).

Remark 3.9. Note that every Killing vector field is affine.

Definition 3.10. A proper static reference frame Z_1 on M is called coweakly-affine if P_1 is a geodesic vector field (i.e., $\nabla_{P_1} P_1 = 0$) and

$$\langle (\mathcal{L}_{P_1} \nabla)(U, V), V\rangle = 0 \quad \text{for every } U, V \perp P_1.$$

Proposition 3.11. *Let Z_1 be a proper static reference frame on M. If P_1 is a geodesic vector field and $\langle (\mathcal{L}_{P_1} \nabla)(X, Y), Y\rangle = 0$ for every $X, Y \in \Gamma W_2$ then Z_1 is a coweakly-affine static reference frame.*

PROOF. First note that, since $\mathcal{L}_{P_1} Z = 0$ and $\nabla_Z Z = f\|\nabla f\| P_1$,

$$\begin{aligned}\langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Z\rangle &= \langle \mathcal{L}_{P_1} \nabla_Z Z, Z\rangle - \langle \nabla_Z \mathcal{L}_{P_1} Z, Z\rangle \\ &\quad - \langle \nabla_{\mathcal{L}_{P_1} Z} Z, Z\rangle = 0.\end{aligned}$$

Also note that since $\mathcal{L}_{P_1} \nabla$ is tensorial, it suffices to check

$$\langle (\mathcal{L}_{P_1} \nabla)(U, V), V\rangle = 0 \quad \text{only for } U = Z + X \text{ and } V = Z + Y,$$

where $X, Y \in \Gamma W_2$ are Lie parallel vector fields along P_1. (Note that since P_1 is a geodesic vector field and W_2 is integrable, such $X, Y \in W_2$ can always be constructed locally). Then

$$\begin{aligned}&\langle (\mathcal{L}_{P_1} \nabla)(Z + X, Z + Y), Z + Y\rangle \\ &\quad = \langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Z\rangle + \langle (\mathcal{L}_{P_1} \nabla)(Z, Y), Z\rangle \\ &\qquad + \langle (\mathcal{L}_{P_1} \nabla)(X, Z), Z\rangle + \langle (\mathcal{L}_{P_1} \nabla)(X, Y), Z\rangle \\ &\qquad + \langle (\mathcal{L}_{P_1} \nabla)(Z, Z), Y\rangle + \langle (\mathcal{L}_{P_1} \nabla)(Z, Y), Y\rangle \\ &\qquad + \langle (\mathcal{L}_{P_1} \nabla)(X, Z), Y\rangle + \langle (\mathcal{L}_{P_1} \nabla)(X, Y), Y\rangle.\end{aligned}$$

By the assumption, the first and the last terms vanish. Others can easily be shown to vanish. For example, since $\mathcal{L}_{P_1} Z = \mathcal{L}_{P_1} X = \mathcal{L}_{P_1} Y = 0$,

$$\begin{aligned}\langle(\mathcal{L}_{P_1}\nabla)(X,Z),Y\rangle &= \langle\mathcal{L}_{P_1}\nabla_X Z - \nabla_X\mathcal{L}_{P_1}Z - \nabla_{\mathcal{L}_{P_1}X}Z, Y\rangle\\ &= \langle\mathcal{L}_{P_1}\nabla_X Z, Y\rangle = 0;\end{aligned}$$

since $\nabla_X Z$=0 by the fact that $\langle\nabla_X Z, Z\rangle$= $-$ $\langle\nabla_Z Z, X\rangle$=0, $\langle\nabla_X Z, P_1\rangle$= $-\langle Z, \nabla_X P_1\rangle = 0$ (since $Z^\perp$ is totally geodesic) and $\langle\nabla_X Z, Y\rangle =$ $-\langle Z, \nabla_X Y\rangle = \langle\nabla_Y Z, X\rangle = -\langle\nabla_X Z, Y\rangle = 0$ for every $Y \in \Gamma W_2$.

Also,

$$\begin{aligned}\langle(\mathcal{L}_{P_1}\nabla)(Z,Y),Y\rangle &= \langle\mathcal{L}_{P_1}\nabla_Z Y - \nabla_Z\mathcal{L}_{P_1}Y - \nabla_{\mathcal{L}_{P_1}Z}Y, Y\rangle\\ &= \langle\mathcal{L}_{P_1}\nabla_Z Y, Y\rangle = \langle\mathcal{L}_{P_1}\nabla_Y Z, Y\rangle + \langle\mathcal{L}_{P_1}\mathcal{L}_Z Y, Y\rangle\\ &= \langle\mathcal{L}_{P_1}\mathcal{L}_Z Y, Y\rangle = 0\end{aligned}$$

since $\mathcal{L}_{P_1}\mathcal{L}_Z Y = [P_1,[Z,Y]] = -[Z,[Y,P_1]] - [Y,[P_1,Z]] = 0$ by the Jacobi identity. □

Lemma 3.12. *Let Z_1 be a coweakly-affine static reference frame. Then*

$$\langle\nabla_U\nabla_V P_1, V\rangle = \langle\nabla_{\nabla_U V} P_1, V\rangle$$

for every $U, V \perp P_1$.

Proof. Let R be the curvature tensor. Then

$$\begin{aligned}R(U,P_1)V &= \nabla_U\nabla_{P_1}V - \nabla_{P_1}\nabla_U V - \nabla_{[U,P_1]}V\\ &= \nabla_U\nabla_V P_1 + \nabla_U\mathcal{L}_{P_1}V - \nabla_{\nabla_U V}P_1 - \mathcal{L}_{P_1}\nabla_U V + \nabla_{\mathcal{L}_{P_1}U}V\\ &= \nabla_U\nabla_V P_1 - \nabla_{\nabla_U V}P_1 - (\mathcal{L}_{P_1}\nabla)(U,V).\end{aligned}$$

Hence

$$\begin{aligned}0 = \langle R(U,P_1)V, V\rangle &= \langle\nabla_U\nabla_V P_1, V\rangle - \langle\nabla_{\nabla_U V}P_1, V\rangle\\ &\quad - \langle(\mathcal{L}_{P_1}\nabla)(U,V), V\rangle.\end{aligned}$$

Thus $\langle\nabla_U\nabla_V P_1, V\rangle = \langle\nabla_{\nabla_U V}P_1, V\rangle$. □

Definition 3.13. Let Z_1 be a coweakly-affine static reference frame on M. Then the shape operator $L_{P_1} : W_2 \mapsto W_2$ of W_2 is defined by $L_{P_1}X = -\nabla_X P_1$. Also the extrinsic curvature function $\kappa_i : M \mapsto \mathbb{R}$ of W_i, $(i = 1, 2)$, is defined by

$$\kappa_i(p) = \frac{\langle R(x_i,y_i)y_i, x_i\rangle}{\langle x_i,x_i\rangle\langle y_i,y_i\rangle - \langle x_i,y_i\rangle^2}\,,$$

where $x_i, y_i \in W_{i_p}$ with $\langle x_i, x_i\rangle\langle y_i, y_i\rangle$ $-\langle x_i, y_i\rangle^2 \neq 0$.

Remark 3.14. Note that L_{P_1} is well-defined since $\langle \nabla_X P_1, Z\rangle = -\langle P_1, \nabla_X Z\rangle = 0$ and $\langle \nabla_X P_1, P_1\rangle = \frac{1}{2} X\langle P_1, P_1\rangle = 0$ for every $X \in W_2$. Also κ_i is well-defined since it corresponds to the curvatures of the planes in W_i, $i = 1, 2$.

Definition 3.15. Let Z_1 be a coweakly-affine static reference frame on M. Z_1 is called spherical if $\det L_{P_1} \geq 0$ and $\kappa_i > 0$, $i = 1, 2$.

Remark 3.16. Let Z_1 be a spherical coweakly affine static reference frame. The sphericality of Z_1 can physically be interpreted as the shape of a star must be "extrinsically and intrinsically" spherical and radially atractive rather than repulsive. The "nonrotationality" of the star is essentially described by that Z_1 is a coweakly-affine static reference frame. Here, note that every Killing field is an affine vector field. Thus, since $P_1 = \frac{\nabla f}{\|\nabla f\|}$ and is weakly-affine, P_1 can also be considered as a "weakly-static" vector field. Hence by assuming that Z_1 is coweakly-affine static, actually we are imposing some staticity on the distribution $W_1 = \operatorname{span}\{Z_1, P_1\}$. But the "staticity" of W_1 should be considered as an "affine staticity", that is, which does not involve the curvature tensor.

Now we can give a local characterization of a "spherical static star".

Theorem 3.17. *Let Z_1 be a spherical coweakly-affine static reference frame on M. Then M is locally a warped product $(M_1 \times_{\psi^2} M_2, \langle\, ,\,\rangle_1 \oplus \psi^2\langle\, ,\,\rangle_2)$, where $(M_1, \langle\, ,\,\rangle_1)$ and $(M_2, \langle\, ,\,\rangle_2)$ are respectively, Lorentzian and Riemannian surfaces with positive curvature. Furthermore if $\nabla\kappa_2 \perp W_2$, then $(M_2, \langle\, ,\,\rangle_2)$ is of constant positive curvature.*

Proof. Note that by Lemmas 3.6 and 3.7, W_1 and W_2 are integrable. First we will show that the integral manifolds of W_1 are totally geodesic. Indeed, it suffices to show that $\nabla_Z Z \in \Gamma W_2$, $\nabla_{P_1} P_1 \in \Gamma W_2$ and $\nabla_{P_1} Z \in \Gamma W_2$. The first two statements follow immediately since $\nabla_Z Z = f\|\nabla f\| P_1$ and $\nabla_{P_1} P_1 = 0$. For the last one, let $X \in \Gamma W_2$, then since

$$\langle \nabla_{P_1} Z, X\rangle = -\langle \nabla_X Z, P_1\rangle = \langle Z, \nabla_X P_1\rangle = 0$$

by the Remark 3.14, it follows that $\nabla_{P_1} Z \in \Gamma W_1$. Next we will show that the integral manifolds of W_2 are totally umbilic. Let $X, Y \in \Gamma W_2$ and $\mathbb{I}$ be the 2nd fundamental form tensor of the integral manifolds of W_2.

Note that since $\langle \nabla_X Y, Z\rangle = -\langle \nabla_X Z, Y\rangle = 0$ by the Remark 3.14,

$$\mathbb{I}(X, Y) = \langle \nabla_X Y, P_1\rangle P_1 = -\langle Y, \nabla_X P_1\rangle P_1.$$

Hence, if T and $\perp$ denote the components tangent and orthogonal to W_2 respectively, and $\nabla^\perp$ is the normal connection,

$$\begin{aligned}(\nabla_X \mathbb{I})(Y,Y) &= \nabla^\perp_X \mathbb{I}(Y,Y) - 2\mathbb{I}((\nabla_X Y)^T, Y)\\ &= -X\langle Y, \nabla_Y P_1\rangle + \langle Y, \nabla_Y P_1\rangle (\nabla_X P)^\perp + 2\langle Y, \nabla_{(\nabla_X Y)^T} P_1\rangle P_1\\ &= \langle \nabla_X Y, \nabla_X P_1\rangle P_1 - \langle Y, \nabla_X \nabla_Y P_1\rangle P_1\\ &= \langle Y, \nabla_{\nabla_X Y} P_1\rangle P_1 - \langle Y, \nabla_X \nabla_Y P_1\rangle P_1\\ &= \langle Y, \nabla_{\nabla_X Y} P_1 - \nabla_X \nabla_Y P_1\rangle P_1\\ &= 0 \quad \text{by Lemma 3.12.}\end{aligned}$$

Thus, since $(\nabla_X \mathbb{I})$ is symmetric, it follows that $(\nabla_X \mathbb{I}) = 0$.

This has two implications: First, from the Codazzi equation (cf. [8, pag. 115]), $R(X,Y)V \in \Gamma W_2$ for every $X, Y, V \in \Gamma W_2$. Second, if c_i, $i = 1, 2$ are the eigenvalues of L_{P_1} then $\nabla c_i \perp W_2$, that is, c_i is constant along the integral manifolds of W_2. Now we will show that $c_1 = c_2$. Indeed, let X_1, X_2 be orthonormal eigenvectors of L_{P_1} corresponding to c_1 and c_2 respectively. Then, since

$$\langle \nabla_{\nabla_X Y} P_1, Y\rangle = \langle \nabla_Y P_1, \nabla_X Y\rangle$$

and

$$\langle \nabla_X \nabla_Y P_1, Y\rangle = X\langle \nabla_Y P_1, Y\rangle - \langle \nabla_Y P_1, \nabla_X Y\rangle,$$

Lemma 3.12 can also be written as

$$X\langle \nabla_Y P_1, Y\rangle = 2\langle \nabla_Y P_1, \nabla_X Y\rangle$$

for $X, Y \in \Gamma W_2$. Hence, by setting $Y = X_1 - X_2$, since $X\langle \nabla_Y P_1, Y\rangle = X(c_1 + c_2) = 0$, we obtain

$$0 = 2\langle \nabla_Y P_1, \nabla_Y Y\rangle = 2(c_2 - c_1)\langle X_1, \nabla_X X_2\rangle$$

for every $X \in \Gamma W_2$. But if $c_2 \neq c_1$, then $(\nabla_X X_i)^T = 0$ $(i = 1, 2)$ for every $X \in \Gamma W_2$ and hence the curvature tensor R_2 of the induced Riemannian structure of the integral manifolds of W_2 is identically zero. But then from the Gauss equations

$$\begin{aligned}0 = \langle R_2(X_1, X_2)X_2, X_1\rangle &= \langle R(X_1, X_2)X_2, X_1\rangle\\ &+ \langle L_{P_1} X_1, X_1\rangle\langle L_{P_1} X_2, X_2\rangle = \kappa_2 + c_1 c_2 = \kappa_2 + det\, L_{P_1}\end{aligned}$$

in contradiction with the assumption that Z_1 is spherical. Thus $c_1 = c_2 = c$ and it follows that $\mathbb{I}(X,Y) = \langle X, Y\rangle P$, where $P = cP_1$. Hence the integral manifolds of W_2 are totally umbilic with normal parallel normal curvature vector field P. Then it follows from [9, Prop. 3(c)] that M is locally a warped product $M_1 \times_{\psi^2} M_2$ with M_1 of curvature $\kappa_1 > 0$ and M_2 is of constant curvature $\kappa_2 + c^2 > 0$. Furthermore, if κ_2 is constant along the integral submanifolds of W_2 then M_2 is of constant curvature. $\square$

Remark 3.18. Note that the local warping function g^2 in $I_{g^2} \times N$ may be quite different from the warping function ψ^2 in $M_1 \times_{\psi^2} M_2$. For example in the Schwarzschild metric in usual coordinates, $g^2 = (1 - \frac{2m}{r})$ and $\psi^2 = r^2$. Also we can introduce Schwarzschild type coordinates for M_1. Note that since $[P_1, Z] = 0$, there exists a chart (t, r) such that $Z = \partial/\partial t$ and $P_1 = \partial/\partial r$. Also since $Zf = 0$, where $\langle Z, Z\rangle = -f^2$, f is only a function of r. Hence the metric is locally of the form $-f^2(r)dt \otimes dt + dr \otimes dr$ on M_1. Also note that since $[hP_1, Z] = 0$ for any function h depending only on r, one may introduce other Schwarzschild type coordinates for M_1. Furthermore, by [8, Prop. 35, pag. 206], since $\mathbb{I}(X, Y) = \langle X, Y\rangle P = -\langle X, Y\rangle(\frac{\nabla\psi}{\psi})$, $\nabla\psi$ is proportional to P and hence $\langle\nabla\psi, Z\rangle = 0$. Thus ψ is only a function of r. That is, the 2 by 2 warped metric above is locally a spherically symmetric static metric [7, pag. 594]

Remark 3.19. Static parts of the Schwarzschild and Reissner metrics are examples to the above theorem which describes the metric of the static part of a "static star". Yet Schwarzschild and Reissner metrics are in fact, special cases of the above theorem as being infinitesimally isotropic.

Definition 3.20. Let Z_1 be a spherical coweakly-affine static reference frame on a spacetime M obeying the Einstein equation for a stress-energy tensor T with $\operatorname{tr} T = 0$. Z_1 is called symmetric with respect to T if $T(Z_1, Z_1) = -T(P_1, P_1)$.

Theorem 3.21. *Let M be a spacetime obeying the Einstein equation for a stress-energy tensor T with $\operatorname{tr} T = 0$. If Z_1 is a spherical coweakly-affine static reference frame on M and is symmetric with respect to T, then M is locally a warped product given in Theorem 3.17, and furthermore:*

(a) $R(z, x)y = \mu\langle x, y\rangle z$ *for every* $z \in W_1$, $x, y \in W_2$ *and viceversa, where* $\mu < 0$ *is a smooth function.*

(b) $R(x, y)z = \kappa_i R_0(x, y)z$ *for every* $x, y, z \in W_i$, $i = 1, 2$, *where* $R_0(x, y)z = \langle z, y\rangle x - \langle x, z\rangle y$.

(c) $T = \eta(-\langle\,,\,\rangle_1 \oplus \langle\,,\,\rangle_2)$, *where* $\eta = 2\mu + \kappa_2 = -2\mu - \kappa_1 \geq 0$.

Proof. By Theorem 3.17, M is locally a warped product. Now by using [8, Prop. 7.35, 7.42 and 7.43],

(1) $R(x,y)z = K_1 R_0(x,y)z$ for every $x,y,z \in W_1$, since W_1 is totally geodesic.

(2) $R(x,y)z = (K_2 + \langle P,P\rangle)R_0(x,y)z - \langle P,P\rangle R_0(x,y)z = K_2 R_0(x,y)z$ by the Gauss equations.

(3) Note that since $P_1 = -\frac{\nabla\psi}{c\psi}$ and is a geodesic vector field, $\langle \nabla_{P_1}\nabla\psi, Z_1\rangle = 0$. Hence Z_1 and P_1 are the eigenvectors of the Hessian tensor $\nabla\nabla\psi$, corresponding to the eigenvalues, say a and b, respectively. Then, since

$$Ric(U,V) = \kappa_1\langle U,V\rangle - \frac{2}{\psi}\langle \nabla_U\nabla\psi, V\rangle$$

for every $U,V \in \Gamma W_1$ and $T = Ric$, it follows that

$$Ric(Z_1,Z_1) = -\kappa_1 + \frac{2}{\psi}a = -Ric(P_1,P_1) = -\kappa_1 + \frac{2}{\psi}b.$$

Thus $a = b = -\psi\mu$ for some function μ and hence $\nabla_U\nabla\psi = -\psi\mu U$ for every $U \in \Gamma W_1$.

Then, for any $U \in \Gamma W_1$ and $X,Y \in \Gamma W_2$,

$$R(U,X)Y = -\frac{\langle X,Y\rangle}{\psi}\nabla_U\nabla\psi = \mu\langle X,Y\rangle U$$

and for any $X \in \Gamma W_2$ and $U,V \in \Gamma W_1$

$$R(X,U)V = -\frac{\langle \nabla_U\nabla\psi, V\rangle}{\psi}X = \mu\langle U,V\rangle X.$$

Also, since $0 \leq T(Z_1,Z_1) = Ric(Z_1,Z_1) = -\kappa_1 - 2\mu$ and $\kappa_1 > 0$, it follows that $\mu < 0$. Hence we showed (a) and (b), that is M is infinitesimally isotropic with respect to $TM = W_1 \oplus W_2$. Then it follows from [2, Prop. 4.4] that

$$T = Ric = (\kappa_1 + 2\mu)\langle\,,\,\rangle_1 \oplus (\kappa_2 + 2\mu)\langle\,,\,\rangle.$$

But $\operatorname{tr} T = 0$, $\kappa_1 + \kappa_2 + 4\mu = 0$, and hence $T = \eta(-\langle\,,\,\rangle_1 \oplus \langle\,,\,\rangle_2)$, where $\eta = \kappa_2 + 2\mu = -\kappa_1 - 2\mu \geq 0$. □

Remark 3.22. Note that the stress-energy tensor T in the above theorem corresponds to the electromagnetic stress-energy tensor in the Reissner solution, where η corresponds to negative Faraday stresses (cf. [1, pag. 124]). Indeed, with no matter present but a radial electric field in the "affinely static", $W_1 \oplus W_2$ possesses such a stress-energy tensor, (cf. [7, pag. 840]). Also see [5], [6] for a global version of the above theorem by making asymptotic considerations.

Remark 3.23. A spacetime with a curvature tensor as in Theorem 3.21 is called infinitesimally isotropic (equivalently, null anisotropic) with respect to the decomposition $TM = W_1 \oplus W_2$ (see [2] and [3]). In fact, Theorem 3.21 together with Theorem 3.17 gives the same conclusion as [3, Th. 3.11]. But we note that [3, Th. 3.11] is also applicable to non-static parts of a "spherically symmetric static star" to give a warped product $M_1 \times_{\psi^2} M_2$ locally. Indeed, although there exists no static reference frame in the "black hole" regions of Scharzschild and Reissner spacetimes, the metric is still a warped product $M_1 \times_{\psi^2} M_2$. In other words, [3, Th. 3.11, 3.16] may be considered as a local characterization of "spherically symmetric static stars".

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