Analytic solutions of Schröder equation

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Let E be a Banach space over the field of complex numbers $\mathbf{C}$.

Definition 1. ([1]) A mapping $f; E\to E$ is called a homogeneous polynomial of degree n if and only if there exists a n-linear symmetrical mapping $\bar{f}$: $E^n\to E$ such that

$$f(x)=\bar{f}(x, \dots, x), \quad x\in E.$$

We say that $\bar{f}$ is associated with f.

Let $P_n(E)$ be a space all continuous and homogeneous polynomials of E into E of degree n $(P_0(E)$ denotes the set of constant mappings from E into $E)$.

We shall investigate local analytic solutions of the Schröder equation

$$\varphi[f(x)] = A\varphi(x), \tag{1}$$

where a function f of the type f: $E\to E$ and $A\in P_1(E)$ are given and φ of the type φ: $E\to E$ is unknown function.

We assume that

(I) The function f is analytic in open set $G\subset E$, $0\in G$, and $f(0)=0$.

(II) The mapping $A:=f'(0)$ is invertible of the space E onto E and

$$\|A\|^2\|A^{-1}\| < 1. \tag{2}$$

Definition 2. ([1]) *A* function f: $G\to E$ is called analytic in G if for every point $x\in G$ there exist $f_n\in P_n(E)$, $n=0, 1, 2, \dots$, such that

$$f(x+h) = \sum_{n=0}^{\infty} f_n(h)$$

for all h in a neighbourhood of $0\in E$.

For every n there exists n-th Frechèt derivative $D^n f(x)$ of f at $x\in G$ and

$$D^n f(x) = n!\,\bar{f}_n, \quad \text{for} \quad n = 0, 1, 2, \dots,$$

where $\bar{f}_n$ is associated with f (cf. [1]).

It follows by $A=A^{-1}AA$ and (2) that

$$s := \|A\| \leqq \|A^{-1}\|\,\|A\|^2 < 1. \tag{3}$$

According to (2) there exists a number σ, $s<\sigma<1$, such that

$$\|A^{-1}\| < \frac{1}{\sigma^2}. \tag{4}$$

Lemma 1. *There exist numbers $R>0$ and $K>0$ such that*

$$\|f(x)-Ax\| \leqq K\|x\|^2 \quad \textit{for} \quad \|x\| \leqq R.$$

PROOF. By virtue of hypothesis (I) and by [2] (Th. 6.2) there exist numbers $M>0$ and R_0, $0<R_0<1$ such that

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{for} \quad \|x\| \leqq R_0,$$

$$\{x\colon \|x\| \leqq R_0\} \subset G$$

and

$$\|f(x)\| \leqq M \quad \text{for} \quad \|x\| \leqq R_0.$$

For $\|x\|<\frac{1}{2} R_0$ we have

$$\|f_n(x)\| \leqq \frac{M}{2^n}, \quad n \geqq 1.$$

Hence

$$\sum_{n=2}^{\infty} \|f_n(x)\| \leqq \frac{1}{2} M \quad \text{for} \quad \|x\| \leqq \frac{1}{2} R_0.$$

Let R be a fixed number such that

$$0 < R < \frac{1}{2} R_0.$$

We get

$$\sum_{n=2}^{\infty} \|f_n(x)\| = \sum_{n=2}^{\infty} \left(\frac{\|x\|}{R}\right)^n \left\| f_n\left(\frac{x}{\|x\|} R\right)\right\| \leqq \frac{\|x\|^2}{2R^2} M$$

for $\|x\| \leqq R$. Thus

$$\|f(x)-Ax\| \leqq \frac{M}{2R^2} \|x\|^2 \quad \text{for} \quad \|x\| \leqq R.$$

Putting $K:=\frac{M}{2R^2}$ we get the assertion of this lemma.

Lemma 2. *There exist positive numbers r and ϑ, $s<\vartheta<\sigma$, such that*

$$\|f(x)\| < \vartheta\|x\| \quad \textit{for} \quad \|x\| \leqq r. \tag{5}$$

PROOF. We have by Lemma 1

$$\|f(x)\| \leqq \|f(x)-Ax\| + \|Ax\| \leqq K\|x\|^2 + s\|x\| = (K\|x\| + s)\|x\|.$$

Hence, it follows that there exists sufficiently small number $r>0$ such that inequality (5) is fulfilled for $\|x\| \leqq r$.

In virtue of (5) the function f transforms the ball $\{x\colon \|x\|<r\}$ into itself, so we may define n-th iterate of the function f:

$$f^0(x) = x, \quad f^{n+1}(x) = f(f^n(x)), \quad n = 0, 1, 2, \ldots.$$

The functions f^n, are analytic (cf. [2]) for $\|x\|<r$.

Theorem 1. *If hypotheses* (I) *and* (II) *are fulfilled, then the sequence* $\{A^{-n}f^n(x)\}$ *is uniformly convergent in some neighbourhood of zero and the function*

$$\varphi(x) := \lim_{n\to\infty} A^{-n}f^n(x) \tag{6}$$

is the analytic solution of Schröder equation (1).

PROOF. By lemmas and by (4) we have

$$\|A^{-(n+1)}f^{n+1}(x)-A^{-n}f^n(x)\| = \|A^{-(n+1)}\big(f^{n+1}(x)-Af^n(x)\big)\| \leqq$$
$$\leqq \|A^{-1}\|^{n+1}\|f^{n+1}(x)-Af^n(x)\| \leqq \frac{K}{\sigma^{2(n+1)}}\|f^n(x)\|^2 \leqq \frac{K}{\sigma^2}r^2\left(\frac{\vartheta}{\sigma}\right)^{2n}$$

for $\|x\|\leqq r$. Hence, the sequence $\{A^{-n}f^n(x)\}$ is uniformly convergent in the set $\{x\colon \|x\|\leqq r\}$. The functions $A^{-n}f^n(x)$ are analytic. Thus, the function φ given by (6) is analytic in $\{x\colon \|x\|<r\}$ (cf. [1]). Since

$$\varphi[f(x)] = \lim_{n\to\infty} A^{-n}f^{n+1}(x) = \lim_{n\to\infty} A^{-(n+1)}Af^{n+1}(x) = A\varphi(x),$$

thus φ is the solution of (1). The proof is completed.

It is known that, if $\|A\|<1$, that the mapping $A-I\big((A-I)(x)=Ax-x\big)$ is invertible. Therefore if φ is a solution of equation (1), then $\varphi(0)=0$.

Theorem 2. *If hypotheses* (I) *and* (II) *are fulfilled, then the solution of Schröder equation given by* (6) *is invertible in an neighbourhood of zero.*

PROOF. At first, we shall prove that the sequence

$$\varphi_n'(x) = (A^{-n}f^n)'(x)$$

is uniformly convergent in some neighbourhood of zero. We have

$$\|\varphi_{n+1}'(x)-\varphi_n'(x)\| = \|A^{-(n+1)}f'\big(f^n(x)\big)\circ\ldots\circ f'(x)-A^{-n}f'\big(f^{n-1}(x)\big)\circ\ldots\circ f'(x)\|$$
$$\leqq \|A^{-(n+1)}\|\,\Big\|\big(f'(f^n(x))-A\big)\big(f'(f^{n-1}(x))\circ\ldots\circ f'(x)\big)\Big\| \leqq$$
$$\leqq \|A^{-1}\|^{n+1}\Big\|f'\big(f^n(x)\big)-f'(0)\Big\|\Big\|f'\big(f^{n-1}(x)\big)\Big\|\ldots\|f'(x)\|$$

for $\|x\|\leqq r$, where r is defined in Lemma 2. There exists a number $r_1\leqq r$ such that

$$\|f'(x)\| < \vartheta \quad \text{for} \quad \|x\| \leqq r_1,$$

where ϑ is the constant from Lemma 2. The function $f''(x)$ is analytic therefore (cf. [2]) there exist positive numbers M and $r_2\leqq r_1$ such that

$$\|f''(x)\| \leqq M \quad \text{for} \quad \|x\| \leqq r_2.$$

By mean value theorem we get

$$\Big\|f'\big(f^n(x)\big)-f'(0)\Big\| \leqq M\vartheta^n\|x\| \quad \text{for} \quad \|x\| \leqq r_2.$$

Thus, we have

$$\|\varphi_{n+1}'(x)-\varphi_n'(x)\| \leqq \frac{Mr_2}{\sigma^2}\left(\frac{\vartheta}{\sigma}\right)^{2n}.$$

Hence the uniform convergence of $\{\varphi_n'(x)\}$ for $\|x\| \leqq r_2$ follows. In virtue of Th. 8.6.3 from [3] we have

$$\varphi'(x) = \lim_{n\to\infty} A^{-n} f'(f^{n-1}(x)) \circ \ldots \circ f'(x), \quad \text{and} \quad \varphi'(0) = I.$$

Hence, by Th. 10.2.5. from [3], the function φ is invertible in a neighbourhood of zero.

Theorem 3. *Let hypotheses* (I) *and* (II) *be fulfilled. If function ψ is an analytic solution of Schröder equation* (1) *in a neighbourhood U of zero, then there exists linear and continuous mapping $S\colon E \to E$ such that*

$$SA = AS \quad \text{and} \quad \psi(x) = S\varphi(x),$$

where φ is given by (6).

Proof. We have $\varphi \circ f = A\varphi$, $\psi \circ f = A\psi$, $f = \varphi^{-1} \circ A\varphi$, $f \circ \varphi^{-1} = \varphi^{-1} \circ A$ in U. Let $L\colon = \psi \circ \varphi^{-1}$. Hence

$$AL = A\psi \circ \varphi^{-1} = \psi \circ f \circ \varphi^{-1} = \psi \circ \varphi^{-1} \circ A = L \circ A.$$

Thus, the mapping L is analytic (cf. [4]) and commutable with A. There exists a neighbourhood V of zero such that

$$L(x) = S(x) + \sum_{k=2}^{\infty} S_k(x), \quad x \in V \subset U,$$

where $S_k \in P_k(E)$. We have equality

$$S(A(x)) + \sum_{k=2}^{\infty} S_k(A(x)) = AS(x) + \sum_{k=2}^{\infty} AS_k(x)$$

because $LA = AL$. Hence

$$SA = AS \quad \text{and} \quad S_k A = AS_k, \quad k \geqq 2.$$

From the last equality we get

$$\|S_k\| \leqq \|A^{-1}\| \, \|S_k\| \|A\|^k, \quad k \geqq 2.$$

By virtue of hypothesis (II) and (3), $S_k = 0$ for $k \geqq 2$. Thus $L(x) = S(x)$ and

(7) $$\psi(x) = S\varphi(x) \quad \text{in } V.$$

It follows by identity theorem that (7) holds in U.

Example. Let X be a Banach space over the field of complex numbers **C** and let $E = P_1(X)$ be the Banach space of all continuous linear mappings of X into X. Let ω be an invertible element of E such that $\|\omega\| < 1$, and $D := \sum_{n=0}^{\infty} \|\omega\|^n$. We put $G := \left\{u \in E\colon \|u\| < \frac{1}{D}\right\}$ and

$$f\colon G \ni u \to f(u) = \omega u (I - u)^{-1} \in E,$$

where $I(x)=x$ for every $x\in X$. The above defined function f may be given by formula

$$f(u) = u+u^2+\dots$$

and it is analytic in G (cf. [2] Th. 5.2), $A:=f'(0)=\omega I_E$, $A^{-1}v=\omega^{-1}v$ for $v\in E$. The function

$$\varphi(u) := u[I-(I-\omega)^{-1}u]^{-1} = \sum_{n=1}^{\infty} u\big((I-\omega)^{-1}u\big)^{n-1} \tag{8}$$

is analytic solution of equation

$$\varphi[f(u)] = A\varphi(u)$$

in G. Indeed,

$$\begin{aligned}\varphi[f(u)] &= \omega u(I-u)^{-1}\{I-(I-\omega)^{-1}\omega u(I-u)^{-1}\}^{-1} = \\ &= \omega u(I-u)^{-1}\{(I-\omega)^{-1}[(I-\omega)(I-u)-\omega u](I-u)^{-1}\}^{-1} = \\ &= \omega u(I-u-\omega+\omega u-\omega u)^{-1}(I-\omega) = \omega u(I-u-\omega)^{-1}(I-\omega)\end{aligned}$$

and

$$\begin{aligned}A\varphi(u) &= \omega u[I-(I-\omega)^{-1}u]^{-1} = \omega u\{(I-\omega)^{-1}[(I-\omega)-u]\}^{-1} = \\ &= \omega u(I-\omega-u)^{-1}(I-\omega).\end{aligned}$$

For the mapping $\omega(x)=\omega(x_1,x_2):=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{4}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ we have $\|\omega\| = \sup_{\|x\|=1}\|\omega(x)\| = \frac{1}{2}$, $\|\omega^{-1}\|=4$ and $\|\omega\|^2\|\omega^{-1}\|=1$. Thus inequality (2) in hypothesis (II) is not necessary.

If $\|\omega\|^2\|\omega^{-1}\|<1$ then inequality (2) is fulfilled. In this case, we obtain formula (8) by application formula (6). This solution is invertible in a neighbourhood of zero by Theorem 2 and every analytic solution ψ of equation (1) in a neighbourhood of zero has the form $\psi(u)=S\varphi(u)$, where $S\in P_1(E)$ and $SA=AS$.

References

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(Received January 6, 1976.)