Existence and uniqueness of solutions of a functional-differential equation

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1. *Introduction*

In the present paper we are concerned with the functional-differential equation of the form

$$\varphi'(x) = h\big(x, \varphi(x), \varphi[f(x, \varphi(x))], u\big) \tag{1}$$

with initial condition

$$\varphi(0) = z_0 \tag{2}$$

where φ is an unknown function and h, f are known functions and u is a real parameter.

We shall prove that the problem (1)$-$(2) has exactly one solution defined in the interval $\langle 0, \infty)$ and belonging to a certain function class G, which is defined below and this solution depends continuously on u.

For the equation

$$\varphi'(x) = h\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)], u\big)$$

the corresponding problem has been investigated by author in [3]. The problem of the local existence of solutions of equation (1) has been investigated in [4].

2. *Existence and uniqueness*

In this section we are going to establish a theorem on the existence of a unique solution of the initial-value problem (1)—(2).

We assume the following

HYPOTHESIS 1.

(i) Let $(Y, \|\cdot\|)$ be a Banach space. The functions $h: I\times Y^2\times R\to Y$, $f: I\times Y\to I$ where $I=\langle 0, \infty)$, $R=(-\infty, +\infty)$ are continuous on $I\times Y^2\times R$ and $I\times Y$ respectively.

(ii) There exist continuous functions L_i: $I\to I$, $i=1, 2$ such that for every $z_i, y_i\in Y$, $i=1, 2$, $x\in I$ and $u\in R$ we have

$$\|h(x, z_1, z_2, u)-h(x, y_1, y_2, u)\| \leqq L_1(x)\|z_1-y_1\|+L_2(x)\|z_2-y_2\|.$$

(iii) There exist constants $F \geqq 0$, $r > 1$, $\beta \geqq 0$ such that for every $y, y_1 \in Y$, $x \in I$

$$|f(x, y) - f(x, y_1)| \leqq F \exp(-rL(x)) \|y - y_1\|$$

where

(3) $$L(x) = \int_0^x [L_1(s) + (\beta F + 1) L_2(s)]\, ds, \quad x \geqq 0.$$

(iv) There exist nonnegative constants A, B, C such that for every $y_1, y_2 \in Y$, $x \in I$ and $u \in R$

$$\|h(x, y_1, y_2, u)\| \leqq A \exp(rL(x)) + B\|y_1\| + C\|y_2\|.$$

(v) There exists constant $N \geqq 0$ such that for $x \in I$, $u \in R$

$$\int_0^x \|h(s, 0, 0, u)\|\, ds \leqq N \exp(rL(x)).$$

Let X be the space of all functions $\varphi\colon I \to Y$ which are continuous in I and

(4) $$\sup_{x \in I} (\|\varphi(x)\| \exp(-rL(x))) < \infty.$$

X with the norm (cf. [1])

(5) $$|||\varphi||| = \sup_{x \in I} (\|\varphi(x)\| \exp(-rL(x)))$$

is the Banach space.

We define G as the space of these functions $\varphi \in X$ which fulfil the inequalities

(6) $$\|\varphi(x)\| \leqq \alpha \exp(rL(x)), \quad x \in I,$$

(7) $$\|\varphi(x) - \varphi(z)\| \leqq \beta \exp(rL(x)) |x - z|, \quad x, z \in I, \quad x \geqq z, \quad \alpha, \beta \text{ — constants.}$$

We can verify that G with metric

(8) $$d(\varphi, \psi) = |||\varphi - \psi|||$$

is a complete metric space.

Now we shall prove

Theorem 1. *Suppose that hypothesis* 1 *is fulfilled and let*

(9) $$f(x, y) \leqq x, \quad x \in I, \quad y \in Y.$$

If, moreover, the numbers α, β fulfil the inequalities

(10) $$\alpha \geqq (N + \|z_0\|) r (r-1)^{-1},$$

(11) $$\beta \geqq A + B\alpha + C\alpha$$

then, for every $u \in R$, the initial-value problem (1)—(2) $(z_0 \in Y)$ *has exactly one solution $\varphi \in G$, given as the limit of successive approximations.*

PROOF. Let $u \in R$ be fixed. Equation (1) with initial condition (2) is equivalent with the equation

$$\varphi(x) = z_0 + \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u)\, ds.$$

Now we define the transformation $\Phi = T\varphi$ by the formula

$$\Phi(x) = z_0 + \int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u)\, ds. \tag{12}$$

We shall prove that (12) transforms G into itself. Let $\varphi \in G$. In view of (i) Φ is continuous in I. From (ii), (6), (9), (3), (v) and (10) we obtain

$$\|\Phi(x)\| \leqq \int_0^x \|h(s, \varphi(s), \varphi[f(s, \varphi(s))], u) - h(s, 0, 0, u)\|\, ds +$$

$$+ \int_0^x \|h(s, 0, 0, u)\|\, ds + \|z_0\| \leqq$$

$$\leqq \int_0^x \{L_1(s)\|\varphi(s)\| + L_2(s)\|\varphi[f(s, \varphi(s))]\|\}\, ds + N\exp(rL(x)) + \|z_0\| \leqq$$

$$\leqq \int_0^x \{L_1(s)\alpha\exp(rL(s)) + L_2(s)\alpha\exp(rL[f(s, \varphi(s))])\}\, ds + N\exp(rL(x)) + \|z_0\| \leqq$$

$$\leqq \alpha \int_0^x \{L_1(s) + L_2(s)\}\exp(rL(s))\, ds + N\exp(rL(x)) + \|z_0\| \leqq$$

$$\leqq \frac{\alpha}{r}\int_0^x r\{L_1(s) + (\beta F + 1)L_2(s)\}\exp(rL(s))\, ds + N\exp(rL(sx)) + \|z_0\| \leqq$$

$$\leqq \frac{\alpha}{r}[\exp(rL(x)) - 1] + N\exp(rL(x)) + \|z_0\| \leqq \alpha\exp(rL(x)).$$

Hence Φ fulfils (6). Next, from (iv), (6), (9), (11) for $x \geqq z$ we have

$$\|\Phi(x) - \Phi(z)\| \leqq \left\|\int_0^x h(s, \varphi(s), \varphi[f(s, \varphi(s))], u)\, ds\right\| \leqq$$

$$\leqq \int_z^x (A + B\alpha + C\alpha)\exp(rL(s))\, ds \leqq \exp(rL(x)) \int_z^x (A + B\alpha + C\alpha)\, ds \leqq$$

$$\leqq (A + B\alpha + C\alpha)\exp(rL(x))|x - z| \leqq \beta\exp(rL(x))|x - z|$$

and condition (7) is fulfilled.

Now we shall prove that the transformation (12) is a contraction map. Actually, from (ii), (8), (7), (iii), (9), (3) for $\Phi=T\varphi$, $\Psi=T\psi$, where $\varphi, \psi\in G$, we get

$$\|\Phi(x)-\Psi(x)\| \leqq$$

$$\leqq \int_0^x \left\|h(s, \varphi(s), \varphi[f(s, \varphi(s))], u)-h(s, \psi(s), \psi[f(s, \psi(s))], u)\right\| ds \leqq$$

$$\leqq \int_0^x \left\{L_1(s)\|\varphi(s)-\psi(s)\|+L_2(s)\left\|\varphi\left[f(s, \varphi(s))\right]-\psi\left[f(s, \psi(s))\right]\right\|\right\} ds \leqq$$

$$\leqq \int_0^x \left\{L_1(s)\|\varphi(s)-\psi(s)\|+L_2(s)\left(\|\varphi[f(s, \varphi(s))]-\varphi[f(s, \psi(s))]\|+\right.\right.$$

$$\left.\left.+\|\varphi[f(s, \psi(s))]-\psi[f(s, \psi(s))]\|\right)\right\} ds \leqq$$

$$\leqq \int_0^x \left\{L_1(s)\, d(\varphi, \psi)\exp\left(rL(s)\right)+L_2(s)(\beta F+1)\, d(\varphi, \psi)\exp\left(rL(s)\right)\right\} ds =$$

$$=\frac{1}{r} d(\varphi, \psi) \int_0^x r\{L_1(s)+(\beta F+1)L_2(s)\}\exp\left(rL(s)\right) ds =$$

$$=\frac{1}{r} d(\varphi, \psi)\left[\exp\left(rL(x)\right)-1\right].$$

Hence

$$d(\Phi, \Psi) \leqq \frac{1}{r} d(\varphi, \psi).$$

On account of Banachs fixed-point theorem there exist exactly one solution $\varphi\in G$ of the problem (1)—(2), which completes the proof.

3. *Continuous dependence on parameter*

Now we consider the problem of the continuous dependence of solutions of the problem (1)—(2) on parameter u.

We assume the following

HYPOTHESIS 2.

There exist constant M and functions $P\colon I\to I$, $\omega\colon I\to I$ such that $\omega(u)\to 0$ as $u\to 0+$ and

$$\exp\left(-rL(x)\right)\int_0^x P(s)\, ds \leqq M, \quad x\in I.$$

Moreover, for $z_1, z_2\in Y$, $x\in I$, $u_1, u_2\in R$ we have

$$\|h(x, z_1, z_2, u_1)-h(x, z_1, z_2, u_2)\| \leqq P(x)\omega(|u_1-u_2|).$$

We have the following

Theorem 2. *Suppose that hypotheses of theorem* 1 *and hypothesis* 2 *are fulfilled. Then solutions of the problem* (1)—(2) *depends continuously on u.*

Proof. For $\varphi\in G$ we define

$$F(x, \varphi, u) = z_0+\int_0^x h\big(s, \varphi(s), \varphi[f(s, \varphi(s))], u\big)\, ds.$$

Similarly as in the proof of theorem 1, for $\varphi, \psi\in G$ we obtain

$$d[F(x, \varphi, u), F(x, \psi, u)] \leqq \frac{1}{r} d(\varphi, \psi). \tag{13}$$

In view of hypothesis 2 we also have

$$d[F(x, \varphi, u_1), F(x, \varphi, u_2)] \leqq M\omega(|u_1-u_2|). \tag{14}$$

From (13), (14) and applying the Banach's fixed-point principle ([2], theorem 3.1, p. 18) we obtain our assertion.

References

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(Received February 18, 1976.)