Remarks on a functional calculus based on Fourier series

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0. Introduction and notation

In algebras with identity the functional calculus can be defined for rational functions [2]. In Banach algebras with identity it can be extended to analytic functions [4]. Moreover, in certain algebras one can go further. For instance, in commutative B^*-algebras with identity the functional calculus can be defined for continuous functions.

In this note, we extend the functional calculus for some particular algebras to arbitrary functions. The principial tool is the following representation theorem which is included as a special case in [10].

Let $\mathscr{A}$ be a commutative algebra over $\mathbf{C}$ and a T_1-space such that the vector space operations are continuous and the multiplication is separately continuous. Suppose that $(e_\alpha)_{\alpha\in\Gamma}$ is an orthogonal family of nonzero idempotents of $\mathscr{A}$ such that $\{e_\alpha\}_{\alpha\in\Gamma}$ has no proper annihilators in $\mathscr{A}$ and $\mathscr{A}*e_\alpha\subset\mathbf{C}e_\alpha$ for all $\alpha\in\Gamma$. Denote $\mathfrak{M}=\mathfrak{M}(\mathscr{A})$ the multiplier extension of $\mathscr{A}$ [7, 9]. Then, for each $F\in\mathfrak{M}$, there exists a unique function $\hat{F}\colon \Gamma\to\mathbf{C}$ such that

$$F=\sum_{\alpha\in\Gamma}\hat{F}(\alpha)e_\alpha$$

in $\mathfrak{M}$. Moreover, the mapping $F\to\hat{F}$ is an algebraic and topological isomorphism of $\mathfrak{M}$ onto $\mathbf{C}^\Gamma$.

Examples for such particular algebras can be found in [10] and also in [1, 3, 5, 6, 8]. The above notations will be used throughout this note without further references.

1. The definition and some basic properties

Definition 1.1. For $\Phi\colon D\subset\mathbf{C}\to\mathbf{C}$, let

$$\tilde{D}=\{F\in\mathfrak{M}\colon \hat{F}(\Gamma)\subset D\}$$

and $\tilde{\Phi}\colon \tilde{D}\to\mathfrak{M}$ such that

$$\tilde{\Phi}(F)=\sum_{\alpha\in\Gamma}\Phi(\hat{F}(\alpha))e_\alpha$$

in $\mathfrak{M}$ for all $F\in\tilde{D}$.

Remark 1.2. By the representation theorem stated in the introduction, it is clear that the above definition is correct. Moreover, since $\hat{z}(\alpha)=z$ for all $z\in\mathbf{C}$ and $\alpha\in\Gamma$, it is also clear that $\tilde{\Phi}$ is an extension of Φ. Therefore, if no confusion seems possible, we may write Φ in place of $\tilde{\Phi}$.

Theorem 1.3. *Let $D \subset \mathbf{C}$. Then the mapping defined on $\mathbf{C}^D$ by*

$$\Phi \to \tilde{\Phi}$$

is an algebraic and topological isomorphism of $\mathbf{C}^D$ into $\mathfrak{M}^{\tilde{D}}$ such that if $\Phi_0(z)=1$ and $\Phi_1(z)=z$ for all $z \in D$, then $\tilde{\Phi}_0(F)=1$ and $\tilde{\Phi}_1(F)=F$ for all $F \in \tilde{D}$.

Proof. Everything stated here is clear, except perhaps the continuity statements. For this, suppose that (Φ_ν) is a net in $\mathbf{C}^D$ and $\Phi \in \mathbf{C}^D$. If $\lim_\nu \Phi_\nu = \Phi$, then we have

$$\lim_\nu \tilde{\Phi}_\nu(F)^\wedge(\alpha) = \lim_\nu \Phi_\nu(\hat{F}(\alpha)) = \Phi(\hat{F}(\alpha)) = \tilde{\Phi}(F)^\wedge(\alpha)$$

for all $F \in \tilde{D}$ and $\alpha \in \Gamma$. Hence, it follows that $\lim_\nu \tilde{\Phi}_\nu(F) = \tilde{\Phi}(F)$ for all $F \in \tilde{D}$, i.e., $\lim_\nu \tilde{\Phi}_\nu = \tilde{\Phi}$. This shows that the mapping $\Phi \to \tilde{\Phi}$ is continuous. To prove that the inverse of this mapping is also continuous, observe that if $\lim_\nu \tilde{\Phi}_\nu = \tilde{\Phi}$, then we have

$$\lim_\nu \Phi_\nu(z) = \lim_\nu \tilde{\Phi}_\nu(z) = \tilde{\Phi}(z) = \Phi(z)$$

for all $z \in D$, i.e., $\lim_\nu \Phi_\nu = \Phi$.

Corollary 1.4. *Let $z_0 \in \mathbf{C}$, $0 < r \leqq +\infty$ and $D = \{z \in \mathbf{C}: |z - z_0| < r\}$. Suppose that $\Phi: D \to \mathbf{C}$ is analytic. Then*

$$\tilde{\Phi}(F) = \sum_{n=0}^{\infty} \frac{\Phi^{(n)}(z_0)}{n!} (F - z_0)^n$$

in $\mathfrak{M}$ for all $F \in \tilde{D}$.

Proof. Define the functions S_n on D by

$$S_n(z) = \sum_{k=0}^{n} \frac{\Phi^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Then a direct application of Theorem 1.3 shows that

$$\tilde{S}_n(F) = \sum_{k=0}^{n} \frac{\Phi^{(k)}(z_0)}{k!} (F - z_0)^k,$$

and $\lim_{n \to \infty} \tilde{S}_n(F) = \tilde{\Phi}(F)$ for all $F \in \tilde{D}$.

Definition 1.5. Let $\mathscr{I}$ be the set of all invertible elements of $\mathfrak{M}$. For $F \in \mathfrak{M}$, the set

$$\sigma(F) = \{\lambda \in \mathbf{C}: F - \lambda \notin \mathscr{I}\}$$

is called the spectrum of F.

Remark 1.6. Observe that we have

$$\sigma(F) = \hat{F}(\Gamma)$$

for all $F \in \mathfrak{M}$.

The following theorem is an elementary version of the spectral mapping theorems [2, 4].

Theorem 1.7. *Let Φ: $D\subset \mathbf{C}\to\mathbf{C}$. Then*

$$\sigma(\tilde{\Phi}(F)) = \Phi(\sigma(F))$$

for all $F\in\tilde{D}$.

Proof. Clearly, we have

$$\sigma(\tilde{\Phi}(F)) = \tilde{\Phi}(F)^\wedge(\Gamma) = \Phi(\hat{F}(\Gamma)) = \Phi(\sigma(F))$$

for all $F\in\tilde{D}$.

Theorem 1.8. *Let Φ: $D\subset\mathbf{C}\to\mathbf{C}$ and Ψ: $E\subset\mathbf{C}\to\mathbf{C}$. Then*

$$(\Phi\circ\Psi)^\sim = \tilde{\Phi}\circ\tilde{\Psi}.$$

Proof. By Theorem 1.7, it is clear that $(\Phi\circ\Psi)^\sim$, and $\tilde{\Phi}\circ\tilde{\Psi}$ have the same domain. On the other hand, if F belongs to the domain of $(\Phi\circ\Psi)^\sim$, then we have

$$(\Phi\circ\Psi)^\sim(F)^\wedge(\alpha) = (\Phi\circ\Psi)(\hat{F}(\alpha)) = \Phi(\Psi(\hat{F}(\alpha))) = \Phi(\tilde{\Psi}(F)^\wedge(\alpha)) =$$
$$= \tilde{\Phi}(\tilde{\Psi}(F))^\wedge(\alpha) = (\tilde{\Phi}\circ\tilde{\Psi})(F)^\wedge(\alpha)$$

for all $\alpha\in\Gamma$, and hence $(\Phi\circ\Psi)^\sim(F)=(\tilde{\Phi}\circ\tilde{\Psi})(F)$.

2. Continuity and differentiation

Theorem 2.1. *Let Φ: $D\subseteq\mathbf{C}\to\mathbf{C}$ and $F_0\in\tilde{D}$. Then $\tilde{\Phi}$ is continuous at F_0 if and only if Φ is continuous at $\hat{F}_0(\alpha)$ for all $\alpha\in\Gamma$.*

Proof. If $\tilde{\Phi}$ is continuous at F_0 and $\alpha\in\Gamma$, then using the function F: $D\to\mathfrak{M}$ defined by

$$F(z) = F_0+(z-\hat{F}_0(\alpha))e_\alpha,$$

we get

$$\lim_{z\to\hat{F}_0(\alpha)}\Phi(z) = \lim_{z\to\hat{F}_0(\alpha)}\Phi(F(z)^\wedge(\alpha)) = \lim_{z\to\hat{F}_0(\alpha)}\tilde{\Phi}(F(z))^\wedge(\alpha) = \tilde{\Phi}(F_0)^\wedge(\alpha) = \Phi(\hat{F}_0(\alpha))$$

and so Φ is continuous at $\hat{F}_0(\alpha)$.

Conversely, if Φ is continuous at $\hat{F}_0(\alpha)$ for all $\alpha\in\Gamma$, then we have

$$\lim_{F\to F_0}\tilde{\Phi}(F)^\wedge(\alpha) = \lim_{F\to F_0}\Phi(\hat{F}(\alpha)) = \Phi(\hat{F}_0(\alpha)) = \tilde{\Phi}(F_0)^\wedge(\alpha)$$

for all $\alpha\in\Gamma$, and hence $\lim_{F\to F_0}\tilde{\Phi}(F)=\tilde{\Phi}(F_0)$, i.e., $\tilde{\Phi}$ is continuous at F_0.

Proposition 2.2. *Let $D\subset\mathbf{C}$ such that $D\neq\mathbf{C}$, and suppose that Γ is not finite. Then $\tilde{D}$ has empty interior in $\mathfrak{M}$.*

Proof. Let $F\in\mathfrak{M}$ and $z\in\mathbf{C}\setminus D$. For each finite subset A of Γ, let $\alpha_A\in\Gamma\setminus A$ and

$$F_A = F+(z-\hat{F}(\alpha_A))e_{\alpha_A},$$

and consider the family of all finite subsets of Γ directed by set inclusion. Then, it is clear that (F_A) is a net in $\mathfrak{M}\setminus\tilde{D}$ such that

$$\lim_A F_A = F,$$

namely $\hat{F}_A(\alpha_A)=z$ for all finite subset A of Γ and $\lim_A \hat{F}_A(\alpha)=\hat{F}(\alpha)$ for all $\alpha\in\Gamma$. Thus, F cannot be an interior point of $\tilde{D}$ in $\mathfrak{M}$.

Corollary 2.3. *If Γ is not finite, then $\mathscr{I}$ has empty interior in $\mathfrak{M}$.*

Proof. This follows immediately from the above proposition, since $\mathscr{I}=(\mathbf{C}\setminus\{0\})^{\sim}$.

Definition 2.4. Let $\Phi\colon D\subset\mathfrak{M}\to\mathfrak{M}$ and $F_0\in D$. If F_0 is a limit point of $(F_0+\mathscr{I})\cap D$ in $\mathfrak{M}$ and the limit

$$\Phi'(F_0) = \lim_{F\to F_0} (F-F_0)^{-1}\big(\Phi(F)-\Phi(F_0)\big)$$

exists in $\mathfrak{M}$, then Φ is said to be differentiable at F_0 and $\Phi'(F_0)$ is called the derivative of Φ at F_0.

Proposition 2.5. Let $D\subset\mathbf{C}$ and $F_0\in\mathfrak{M}$. Then F_0 is a limit point of $(F_0+\mathscr{I})\cap\tilde{D}$ in $\mathfrak{M}$ if and only if $\hat{F}_0(\alpha)$ is a limit point of D in $\mathbf{C}$ for all $\alpha\in\Gamma$.

Proof. If F_0 is a limit point of $(F_0+\mathscr{I})\cap\tilde{D}$, then there exists a net (F_ν) in $(F_0+\mathscr{I})\cap\tilde{D}$ such that $\lim_\nu F_\nu=F_0$. Thus, for each $\alpha\in\Gamma$, $(\hat{F}_\nu(\alpha))$ is a net in $D\setminus\{\hat{F}(\alpha)\}$ such that $\lim_\nu \hat{F}_\nu(\alpha)=\hat{F}_0(\alpha)$, and so $\hat{F}_0(\alpha)$ is a limit point of D.

To prove the converse, suppose that $\hat{F}_0(\alpha)$ is a limit point of D for all $\alpha\in\Gamma$. Then, for each $\alpha\in\Gamma$, there exists a sequence $(z_n(\alpha))$ in $D\setminus\{\hat{F}_0(\alpha)\}$ such that $\lim_{n\to\infty} z_n(\alpha)=\hat{F}_0(\alpha)$. Define

$$F_n = \sum_{\alpha\in\Gamma} z_n(\alpha)e_\alpha.$$

Then, we have $\hat{F}_n(\alpha)=z_n(\alpha)$ for all $\alpha\in\Gamma$. Thus, (F_n) is a sequence in $(F_0+\mathscr{I})\cap\tilde{D}$ such that $\lim_{n\to\infty} F_n=F_0$, and so F_0 is a limit point of $(F_0+\mathscr{I})\cap\tilde{D}$.

Corollary 2.6. *$\mathfrak{M}$ is the derived set of $\mathscr{I}$ in $\mathfrak{M}$.*

Proof. By the above proposition, every $F\in\mathfrak{M}$ is a limit point of $(F+\mathscr{I})\cap\mathscr{I}$, since $\hat{F}(\alpha)$ is a limit point of $\mathbf{C}\setminus\{0\}$ for all $\alpha\in\Gamma$.

Theorem 2.7. *Let $\Phi\colon D\subset\mathbf{C}\to\mathbf{C}$ and $F_0\in\tilde{D}$. Then $\tilde{\Phi}$ is differentiable at F_0 if and only if Φ is differentiable at $\hat{F}(\alpha)$ for all $\alpha\in\Gamma$. Moreover, in this case*

$$(\tilde{\Phi})'(F_0) = (\Phi')^{\sim}(F_0)$$

holds.

Proof. Suppose first that $\tilde{\Phi}$ is differentiable at F_0, and let $\alpha\in\Gamma$. Then, by Proposition 2.5, $\hat{F}_0(\alpha)$ is a limit point of $D=(\hat{F}_0(\alpha)+\mathscr{I})\cap D$. Moreover, using the notation

$$F(z) = F_0+\big(z-\hat{F}_0(\alpha)\big)e_\alpha,$$

we can infer that

$$\lim_{z\to\hat F_0(\alpha)}\big(z-\hat F_0(\alpha)\big)^{-1}\big(\Phi(z)-\Phi(\hat F_0(\alpha))\big)=$$
$$=\lim_{z\to\hat F_0(\alpha)}\big(F(z)^\wedge(\alpha)-\hat F_0(\alpha)\big)^{-1}\big(\Phi(F(z)^\wedge(\alpha))-\Phi(\hat F_0(\alpha))\big)=$$
$$=\lim_{z\to\hat F_0(\alpha)}\big((F(z)-F_0)^{-1}(\tilde\Phi(F(z))-\tilde\Phi(F_0))\big)^\wedge(\alpha)=(\tilde\Phi)'(F_0)^\wedge(\alpha),$$

and so Φ is differentiable at $\hat F_0(\alpha)$.

Suppose now that Φ is differentiable at $\hat F_0(\alpha)$ for all $\alpha\in\Gamma$. Then, by Proposition 2.5, F_0 is a limit point of $(F_0+\mathcal{I})\cap\tilde D$, and moreover,

$$\lim_{F\to F_0}\big((F-F_0)^{-1}(\tilde\Phi(F)-\tilde\Phi(F_0))\big)^\wedge(\alpha)=\lim_{F\to F_0}\big(\hat F(\alpha)-\hat F_0(\alpha)\big)^{-1}\big(\Phi(\hat F(\alpha))-\Phi(\hat F_0(\alpha))\big)=$$
$$=\Phi'\big(\hat F_0(\alpha)\big)$$

for all $\alpha\in\Gamma$, and hence

$$\lim_{F\to F_0}(F-F_0)^{-1}\big(\tilde\Phi(F)-\tilde\Phi(F_0)\big)=\sum_{\alpha\in\Gamma}\Phi'\big(\hat F_0(\alpha)\big)e_\alpha.$$

This shows that $\tilde\Phi$ is differentiable at F_0 and $(\tilde\Phi)'(F_0)=(\Phi')^\sim(F_0)$.

3. The exponential function

Definition 3.1. Let $\exp\colon\mathfrak{M}\to\mathfrak{M}$ such that

$$\exp F=\sum_{\alpha\in\Gamma}\exp\hat F(\alpha)e_\alpha$$

for all $F\in\mathfrak{M}$.

Remark 3.2. It is clear that the above definition is correct, and that this exponential function is an extension of the complex one.

Theorem 3.3. *For every $F\in\mathfrak{M}$,*

$$\exp F=\sum_{n=0}^{\infty}\frac{1}{n!}F^n$$

in $\mathfrak{M}$.

PROOF. This follows immediately from Corollary 1.4.

Theorem 3.4. *The function* exp *is differentiable and*

$$\exp'=\exp.$$

PROOF. This follows immediately from Theorem 2.7.

Theorem 3.5. *For every $F, G\in\mathfrak{M}$,*

$$\exp(F+G)=\exp F*\exp G.$$

PROOF. This follows immediately from the fact that

$$\big(\exp(F+G)\big)^\wedge(\alpha)=\exp\big(\hat F(\alpha)+\hat G(\alpha)\big)=\exp\hat F(\alpha)\exp\hat G(\alpha)=$$
$$=(\exp F)^\wedge(\alpha)(\exp G)^\wedge(\alpha)=(\exp F*\exp G)^\wedge(\alpha)$$

for all $\alpha\in\Gamma$.

Theorem 3.6. *For $F, G \in \mathfrak{M}$, we have* $\exp F = \exp G$ *if and only if there exists* $H \in \mathfrak{M}$ *such that* $\sigma(H) \subset \mathbf{Z}$ *and* $F = G + 2\pi i H$.

PROOF. Clearly, we have $\exp F = \exp G$ if and only if $\exp \hat{F}(\alpha) = \exp \hat{G}(\alpha)$, i.e., $\frac{1}{2\pi i}(\hat{F}(\alpha) - \hat{G}(\alpha)) \in \mathbf{Z}$ for all $\alpha \in \Gamma$. Hence, by Remark 1.6, the assertion is quite obvious.

Theorem 3.7. We have

$$\exp \mathfrak{M} = \mathscr{I}.$$

PROOF. This follows immediately from the facts that $\exp \mathbf{C} = \mathbf{C} \setminus \{0\}$ and $\mathscr{I} = (\mathbf{C} \setminus \{0\})^{\sim}$.

Remark 3.8. A special case of this exponential function can be used to relate translation and differentation in the algebra of periodic generalized functions [6, Remark 4.7].

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