A generalization of the Eulerian numbers

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Summary

In this paper we shall demonstrate that the solution of an occupancy problem leads in a natural way to a generalization of the Eulerian numbers.

1. Introduction

In a series of random trials we distribute balls at random in a set of boxes numbered $1, 2, 3, \ldots$. In the n-th trial we distribute l balls ($l \geqq 1$) in the first n boxes in such a way that we put each ball independently of the others in any one of the n boxes with probability $1/n$. Let $\nu(n)$ denote the number of empty boxes among the first n boxes at the completion of the n-th trial and define $\nu(0)=0$. We are concerned with the determination of the distribution of the random variable $\nu(n)$ for $n=0, 1, 2, \ldots$.

Write

$$\mathbf{P}\{\nu(n) = k\} = P(n, k) \tag{1}$$

for $0 \leqq k \leqq n$ and define

$$B_r(n) = \sum_{k=r}^{n} \binom{k}{r} P(n, k) \tag{2}$$

for $0 \leqq r \leqq n$ as the r-th binomial moment of $\nu(n)$.

If we know $B_r(n)$ for $0 \leqq r \leqq n$, then (1) is given by the formula

$$P(n, k) = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} B_r(n) \tag{3}$$

for $0 \leqq r \leqq n$.

We can easily determine (2). Denote by $A_i(n)$ $(1 \leqq i \leqq n)$ the event that box i is empty at the completion of the n-th trial. Then we have

$$B_r(n) = \sum_{1 \leqq i_1 < i_2 < \ldots < i_r \leqq n} \mathbf{P}\{A_{i_1}(n) A_{i_2}(n) \ldots A_{i_r}(n)\} \tag{4}$$

for $1 \leqq r \leqq n$ and $B_0(n) = 1$. Since now

$$\mathbf{P}\{A_{i_1}(n) A_{i_2}(n) \ldots A_{i_r}(n)\} = [(i_1 - 1)(i_2 - 2) \ldots (i_r - r)(n-r)!/n!]^l \tag{5}$$

for $1 \leqq i_1 < i_2 < \ldots < i_r \leqq n$, therefore (1) is completely determined by (3), (4) and (5).

As an alternative approach we can use the following recurrence formula for the determination of (1):

$$n^l P(n, k) = \sum_{s=0}^{l} P(n-1, k-1+s) \binom{k+s}{k} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (n-k-i)^l \tag{6}$$

for $0 \leqq k \leqq n$. In the sum $P(n-1, j)=0$ if $j \geqq n$ or $j<0$, and $P(0, 0)=1$.

Since in (4) the sum contains $\binom{n}{r}$ terms, the evaluation of $B_r(n)$ becomes unmanageable for large values of n. However, fortunately we can find and explicit expression for $B_r(n)$ in the form of certain generalized Stirling's numbers of the second kind.

2. Generalized Stirling's numbers

For a given positive integer l let us define $\mathfrak{S}(n, j, l)$ for $n \geqq 0$ and $j \geqq 0$ by the recurrence formula

$$\mathfrak{S}(n+1, j, l) = \mathfrak{S}(n, j-1, l) + j^l \mathfrak{S}(n, j, l) \tag{7}$$

where $n \geqq 0$ and $j \geqq 1$ and by the initial conditions $\mathfrak{S}(0, 0, l)=1$, $\mathfrak{S}(n, 0, l)=0$ for $n \geqq 1$ and $\mathfrak{S}(0, j, l)=0$ for $j \geqq 1$. Table 1 contains the numbers $\mathfrak{S}(n, j, l)$ for $n \leqq 4$.

TABLE 1 for $\mathfrak{S}(n, j, l)$.

n \ j	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	2^l+1	1	0
4	0	1	4^l+2^l+1	3^l+2^l+1	1

We have

$$\mathfrak{S}(n, j, l) = \sum_{1 \leqq r_1 \leqq r_2 \leqq \dots \leqq r_{n-j} \leqq j} (r_1 r_2 \dots r_{n-j})^l \tag{8}$$

for $1 \leqq j \leqq n$ and $\mathfrak{S}(n, j, l)=0$ for $0<n<j$. For if $\mathfrak{S}(n, j, l)$ is defined by (8), then (7) is satisfied for $n \geqq 1$ and $\mathfrak{S}(n, 1, l)=1$ for $n \geqq 1$.

From (7) we obtain the generating function

$$\sum_{n=j}^{\infty} \mathfrak{S}(n, j, l) x^n = \prod_{r=1}^{j} \left(\frac{x}{1-r^l x} \right) \tag{9}$$

for $j \geqq 1$ and $|x|<j^{-l}$ and by partial fraction expansion we get the explicit expression

$$\mathfrak{S}(n, j, l) = \sum_{i=0}^{j} (-1)^{j-i} A_i(l) B_{ij}(l) i^{nl} \tag{10}$$

for $0 \leqq j \leqq n$ where

$$A_i(l) = \prod_{0 \leqq r < i} (i^l - r^l)^{-1} \tag{11}$$

for $i \geqq 0$ $(A_0(l)=1)$ and

(12) $$B_{ij}(l) = \prod_{i<r\leqq j} (r^l - i^l)^{-1}$$

for $0 \leqq i \leqq j$ $(B_{jj}(l)=1)$.

If $l=1$, then $\mathfrak{S}(n, j, l) = \mathfrak{S}(n, j)$ where $\mathfrak{S}(n, j)$ $(0 \leqq j \leqq n)$ are Stirling's numbers of the second kind introduced in 1730 by J. STIRLING [59]. See also Ch. JORDAN [34], [35]. If $l=1$ in (10), we get

(13) $$\mathfrak{S}(n, j) = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^n$$

for $0 \leqq j \leqq n$.

If $l=2$, then $\mathfrak{S}(n, j, l) = T(2n, 2n-j)$ where $T(n, k)$ $(0 \leqq k \leqq n)$ are central factorial numbers. See J. LOHNE [41], L. CARLITZ and J. RIORDAN [20], and J. RIORDAN [52, p. 217].

For $l \geqq 1$ the numbers $\mathfrak{S}(n, j, l)$ are particular cases of more general numbers introduced in 1853 by A. CAYLEY [23]. See also O. J. MUNCH [47] and L. COMTET [24].

3. The probability $P(n, k)$

By (5) and (8) we can express (4) in the following explicit form

(14) $$B_r(n) = \mathfrak{S}(n, n-r, l)[(n-r)!]^l/(n!)^l$$

for $0 \leqq r \leqq n$ and thus by (3) we get

(15) $$P(n, k) = A(n, k, l)/(n!)^l$$

for $0 \leqq k \leqq n$ where

(16) $$A(n, k, l) = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} \mathfrak{S}(n, n-r, l)[(n-r)!]^l$$

for $n \geqq 0$, $k \geqq 0$ and $l \geqq 1$. Clearly $A(n, k, l) = 0$ if $k \geqq n \geqq 1$.

Let us define the polynomials

(17) $$A_n(x, l) = \sum_{k=0}^{n} A(n, k, l) x^k$$

for $n \geqq 0$. In particular we have $A_0(x, l) = 1$, $A_1(x, l) = 1$, $A_2(x, l) = 2^l - 1 + x$, $A_3(x, l) = 6^l - 4^l - 2^l + 1 + (4^l + 2^l - 2)x + x^2$, $A_4(x, l) = (24)^l - (18)^l - (12)^l + 8^l - 6^l + 4^l + 2^l - 1 + [(18)^l + (12)^l - 2(8)^l + 6^l - 2(4)^l - 2(2)^l + 3]x + (8^l + 4^l + 2^l - 3)x^2 + x^3$.

In the particular case where $l=1$, let us write $A(n, k, l) = A(n, k)$ and $A_n(x, l) = A_n(x)$. By (10) and (16) we have

(18) $$A(n, k) = A(n, n-1-k) = \sum_{i=0}^{k+1} (-1)^i \binom{n+1}{i} (k+1-i)^n$$

for $0 \leqq k \leqq n-1$ and $A(n, k) = 0$ for $k \geqq n \geqq 1$. Table 2 contains the numbers (18) for $n \leqq 6$.

TABLE 2 for $A(n, k)$

n \ k	0	1	2	3	4	5
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	4	1	0	0	0
4	1	11	11	1	0	0
5	1	26	66	26	1	0
6	1	57	302	302	57	1

The numbers $A(n, k)$ defined by (18) are called Eulerian numbers. It seems L. EULER [30] was the first who studied these numbers in 1736. The numbers $A(n, k, l)$ defined by (16) are some natural generalizations of the numbers $A(n, k)$.

By the results of this paper the numbers $A(n, k, l)$ have a simple combinatorial interpretation. Let us consider a triangular board of size n in which the i-th row contains i cells ($i=1, 2, \ldots, n$). (See Figure 1)

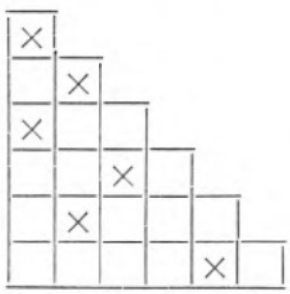

Figure 1

Let us put a mark in a cell in each row of the triangle. We can interpret $A(n, k, l)$ as the number of ways in which we can superimpose l marked triangles such that there are exactly k empty columns.

4. Historical notes

In 1736 L. EULER [30], [31 pp. 373—374] defined the polynomials $A_n(x)=A_n(x, 1)$ as the coefficients in his new summation formula. He determined the explicit form of $A_n(x)$ and proved that

$$\frac{x-1}{x-e^{u(x-1)}} = \sum_{n=0}^{\infty} A_n(x) u^n/n! \tag{19}$$

for $x \neq 1$ and sufficiently small u.

In 1812 P. S. LAPLACE [40 pp. 256—257] demonstrated that if $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent random variables each having a uniform distribution over the interval (0, 1), then

$$\mathbf{P}\{\xi_1+\ldots+\xi_n \leqq x\} = \frac{1}{n!}\sum_{j=0}^{[x]}(-1)^j\binom{n}{j}(x-j)^n \tag{20}$$

for $0 \leqq x \leqq n$. On the other hand if in (15) $l=1$, we have

$$\mathbf{P}\{v(n) \leqq k\} = \mathbf{P}\{v(n) \geqq n-1-k\} = \frac{1}{n!}\sum_{j=0}^{k+1}(-1)^j\binom{n}{j}(k+1-j)^n \tag{21}$$

for $k=0, 1, \ldots, n-1$. A comparison of (20) and (21) shows that

$$\mathbf{P}\{v(n) < k\} = \mathbf{P}\{\xi_1+\ldots+\xi_n \leqq k\} \tag{22}$$

for $k=1, 2, \ldots, n$. For a simple proof of (20) see reference [60]. By (22) it follows that $v(n)$ has the same distribution as $[\xi_1+\ldots+\xi_n]$. This representation of $v(n)$ makes it possible to determine easily the asymptotic behavior of $A(n, k)$ as $n\to\infty$ and $k\to\infty$. For example by the central limit theorem it follows that

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{2v(n)-n}{\sqrt{n/3}} \leqq x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du \tag{23}$$

for any x.

In analysis, number theory, combinatorial analysis and probability theory, several results are connected with Eulerian numbers.

In studying Fermat's last problem in 1857 E. E. KUMMER [39] introduced the polynomials

$$P_n(x, y) = (x+y)^n\left(\frac{d^n \log(x+e^u y)}{du^n}\right)_{u=0} \tag{24}$$

for $n \geqq 1$. It can easily be seen that $P_n(xy, y) = y^n P_n(x, 1)$ and $P_n(x, 1) = xA_{n-1}(-x)$ for $n \geqq 1$. The numbers

$$B_n = nP_n(1, 1)/2^n(2^n-1) \tag{25}$$

defined for $n \geqq 2$ are Bernoulli numbers.

In 1883 J. WORPITZKY [68] demonstrated that the Eulerian numbers (18) satisfy the recurrence formula

$$A(n+1, k) = (k+1)A(n, k)+(n+1-k)A(n, k-1) \tag{26}$$

for $n \geqq 1$ and $k \geqq 1$ where $A(n, 0) = 1$ for $n \geqq 1$ and $A(1, k) = 0$ for $k \geqq 1$. Furthermore, he showed that

$$x^n = \sum_{k=0}^{n-1} A(n, k)\binom{x+k}{n} \tag{27}$$

for $n \geqq 1$. See also H. F. SCHERK [55], E. UNFERDINGER [64], G. FROBENIUS [33], L. KRONECKER [38], L. SAALSCHÜTZ [54], D. MIRIMANOFF [45], and H. S. VANDIVER [65].

In the last quarter of the nineteenth century the works of J. BIENAYMÉ [8] and D. ANDRÉ [1], [2], [3], [4], [5] focused attention on the study of random permutations. Let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a random permutation of $(1, 2, \ldots, n)$ and suppose that every possible permutation has the same probability. In 1879 D. ANDRÉ [1] proved that

$$\mathbf{P}\{\alpha_1 < \alpha_2,\ \alpha_2 > \alpha_3,\ \alpha_3 < \alpha_4, \ldots\} = A(n)/n! \tag{28}$$

for $n \geqq 2$ where

$$\sum_{n=0}^{\infty} A(n) \frac{x^n}{n!} = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sec x + \tan x. \tag{29}$$

In 1908 P. A. MACMAHON [42] by solving a problem of Simon Newcomb concerning a card game proved that the probability that a deck of n cards numbered $1, 2, \ldots, n$ may be dealt into $k+1$ piles, if cards are placed in one pack as long as they are in descending order of magnitude, is given by

$$\mathbf{P}\{\alpha_i < \alpha_{i+1} \text{ for exactly } k \text{ subcripts } i = 1, 2, \ldots, n-1\} A = (n, k)/n! \tag{30}$$

for $k = 0, 1, \cdots, n-1$. The same result has been obtained in various forms by L. SCHRUTKA [56], G. H. MOORE and W. A. WALLIS [46], and J. RIORDAN [50]. See also I. KAPLANSKY and J. RIORDAN [36], R. SPRAGUE [58], L. CARLITZ and J. RIORDAN [19], D. E. BARTON and F. N. DAVID [6], and D. E. BARTON and C. L. MALLOWS [7]. Various generalizations of the problem of Simon Newcomb have been given by P. A. MACMAHON [42], G. KREWERAS [37], L. CARLITZ [14], [15], and J. F. DILLON and D. P. ROSELLE [26].

By the solution of the Smith College diploma problem of R. MAUER [44] we have the following result

$$\mathbf{P}\{\alpha_i < i \text{ for exactly } k \text{ subscripts } i = 1, 2, \ldots, n-1\} = A(n, k)/n! \tag{31}$$

for $k = 0, 1, \ldots, n-1$. See D. WEST [66] and D. FOATA and M.-P. SCHÜTZENBERGER [32].

In 1974 D. DUMONT [27] proved that there are exactly $A(n, k)$ arrangements $(u_1, u_2, \ldots, u_n)$ such that $i \leqq u_i \leqq n$ for $i = 1, 2, \ldots, n$ and $(u_1, u_2, \ldots, u_n)$ contains $k+1$ distinct integers. This interpretation of $A(n, k)$ is in agreement with our interpretation of $A(n, k, l)$ for $l = 1$.

Worpitzky's formula (27) has been rediscovered several times. See e.g. P. S. DWYER [28], [29] and P. A. PIZA [48]. Formula (27) has been generalized by E. B. SHANKS [57] and L. CARLITZ [9].

The asymptotic result (23) has been proved by J. WOLFOWITZ [67], H. B. MANN [43], L. CARLITZ, D. C. KURTZ, R. SCOVILLE and O. P. STACKELBERG [18], and S. TANNY [61].

For the general theory of Eulerian numbers we refer to L. CARLITZ [11], [17], J. RIORDAN [51], and D. FOATA and M.-P. SCHÜTZENBERGER [32]. See also L. TOSCANO [62], [63], L. CARLITZ [13], L. CARLITZ, D. P. ROSELLE and R. A. SCOVILLE [21], D. P. ROSELLE [53] and F. POUSSIN [49].

By generalizing (27) L. CARLITZ [10] introduced the so-called q-Eulerian numbers. Eulerian numbers of higher order have been defined by L. CARLITZ [12]. See also J. F. DILLON and D. P. ROSELLE [25]. In 1973 L. CARLITZ [16] introduced generalized Eulerian numbers by expanding the k-th power of (19) into a Taylor series. See also L. CARLITZ [17], and L. CARLITZ and R. SCOVILLE [22].

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