The use of fractional calculus in obtaining Legendre functions of arbitrary index

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Summary: It is not the purpose of this paper to produce a new result as classical means of obtaining Legendre polynomials of arbitrary index are well-known. But, it is often considered to be of importance that known results be produced by alternative techniques. This paper exemplifies the usefulness of the connection of a derivative of arbitrary order and a special function, and a usefulness of the generalized Leibniz rule for a product. No claim is made that the fractional calculus method is better. However, the little known method of fractional operations has often proved to be useful in simplifying complicated functional equations and deserves a more general recognition and use.

The idea that Legendre polynomials $P_n(z)$ can be defined for arbitrary value of n stems from two sources. One is Murphy's expression for $P_n(z)$ as a hypergeometric function in 1833, and the second is Schläfli's contour integral for $P_n(z)$ in 1881. These details are given in [1].

It will be worthwhile to refresh the reader's memory regarding the commonly used notations of hypergeometric functions.

$$1+\frac{ab}{1!\,g}x+\frac{a(a+1)b(b+1)}{2!\,g(g+1)}x^2+\dots \tag{1}$$

is called a hypergeometric series because it is a generalization of the geometric series $1+x+x^2+\dots$. The notation $(a)_k$, $(b)_k$ and $(g)_k$ are of the form

$$\begin{cases} (a)_k = a(a+1)(a+2)\dots(a+k-1) = \dfrac{\Gamma(a+k)}{\Gamma(a)}, \\ (a)_0 = 1 \quad \text{and} \quad (1)_k = k!\,. \end{cases} \tag{2}$$

The notation

$$_2F_1(a,\, b;\, g;\, x) \tag{3}$$

has the following meaning. The subscript 2 preceding F denotes that there are two parameters of the form (2) and these become factors in the numberator as shown in (4). The subscript l after F denotes one parameter of the form (2). The parameters between the semicolons are denominator factors. Using this notation (1) can be written conveniently in summation form:

$$_2F_1(a,\, b;\, g;\, x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{k!\,(g)_k}x^k. \tag{4}$$

It can also be shown that

$$(5)\qquad (-\nu)_k = \frac{\Gamma(k-\nu)}{\Gamma(-\nu)} = \frac{(-1)^k \Gamma(\nu+1)}{\Gamma(1+\nu-k)}.$$

Murphy's expression for the Legendre function of arbitrary index in terms of (3) is

$$(6)\qquad P_\nu(z) = {}_2F_1(-\nu, (\nu+1); 1; (1-z)/2).$$

Euler's identity is

$$(7)\qquad {}_2F_1(a, b; c; s) = (1-s)^{-a}\,{}_2F_1(a, c-b; c; s/(s-1)).$$

With the use of Euler's identity the right side of (6) can be written as

$$(1-(1-z)/2)^\nu\,{}_2F_1\left(-\nu, -\nu; 1; \frac{(1-z)/2}{(1-z)/2-1}\right),$$

or

$$(8)\qquad \left(\frac{1+z}{2}\right)^\nu {}_2F_1(-\nu, -\nu; 1; (z-1)/(z+1)).$$

In summation form the above is

$$(9)\qquad P_\nu(z) = \frac{1}{2^\nu}\sum_{k=0}^{\infty} \frac{(-\nu)_k(-\nu)_k}{k!\,k!}(z-1)^k(z+1)^{\nu-k},$$

where from (5)

$$(-\nu)_k(-\nu)_k = [\Gamma(\nu+1)/\Gamma(1+\nu-k)]^2.$$

The result (9) seems quite easy and straightforward to obtain by the formal procedure just given. But in all fairness one must admit, on the other hand, that the apparent ease was the result of the use of identity (7) which is neither easy nor straightforward to establish.

Theorem. *Legendre functions of arbitrary index represented by* (9) *may be obtained without the use of Euler's identity* (7). *The form* (9) *may be generated by use of the Riemann—Liouville operators of arbitrary order.*

PROOF. The Riemann—Liouville operator ${}_cD_s^{-\nu}$ for integration of arbitrary order is defined by the definite integral

$$(10)\qquad {}_cD_s^{-\nu}f(s) = \frac{1}{\Gamma(\nu)}\int_c^s (s-t)^{\nu-1}f(t)\,dt, \quad \mathrm{Re}\,(\nu) > 0.$$

When $c=0$ and $c=-\infty$ we have Riemann's and Liouville's definitions respectively. For differentiation of arbitrary order we have

$$(11)\qquad {}_cD_s^\nu f(s) = {}_cD_s^{m-p}f(s) = \frac{d^m}{ds^m}\frac{1}{\Gamma(p)}\int_c^s (s-t)^{p-1}f(t)\,dt,$$

where $\nu>0$, $\nu=m-p$, m is the least integer greater than ν, $0<p\leqq 1$, and d^m/ds^m is the ordinary mth derivative operator.

For a wide class of functions the integral on the right above, when the lower terminal of integration is zero, is a beta integral and is easily evaluated. It can readily be shown that the hypergeometric function ${}_2F_1(a, b; g; s)$ can be expressed by the Riemann—Liouville operator ${}_0D_s^{-v}$. Details are given in [2]. The connection is this:

$$ {}_2F_1(a, b; g; s) = \frac{\Gamma(g)}{\Gamma(b)} s^{1-g} {}_0D_s^{-(g-b)}\left(s^{b-1}(1-s)^{-s}\right), \tag{12} $$

where ${}_0D_s^{-(g-b)}$ denotes the fractional operation of order $(g-b)$ of the product $s^{b-1}(1-s)^{-a}$.

Consider now the hypergeometric function F given in (6) whose parameters are

$$ a=-v, \quad b=v+1, \quad g=1, \quad s=(1-z)/2. $$

Then (12) becomes

$$ P_v(z) = \frac{1}{\Gamma(v+1)} {}_0D_{(1-z)/2}^{v}\left((1-z)/2\right)^v\left(1-(1-z)/2\right)^v. \tag{13} $$

For simplification let $1-z=2u$, and the above becomes

$$ P_v(z) = \frac{1}{\Gamma(v+1)} {}_0D_u^v u^v(1-u)^v. \tag{14} $$

We will now apply the Leibniz rule for the derivative of arbitrary order of a product which is

$$ {}_0D_x^v f(x)g(x) = \sum_{n=0}^{\infty} \binom{v}{n} {}_0D_x^{(n)} f(x) {}_0D_x^{(v-n)} g(x), \tag{15} $$

where $D^{(n)}$ is the ordinary nth differentiation operator d^n/dx^n, $D^{(v-n)}$ is the Riemann fractional operator and $\binom{v}{n}$ is the generalized binomial coefficient $\Gamma(v+1)/n!\,\Gamma(v-n+1)$. In (14) choose u^v to be the function g to which a fractional operation will be applied, and choose $(1-u)^v$ to be the function f to which ordinary differentiation will be applied.

The fractional operations for ${}_0D_u^{v-n}u^v$ for $n=0, 1, 2, \dots, n$ are obtained from (10) and (11):

$$ {}_0D_u^v u^v = \Gamma(v+1), \qquad {}_0D_u^{v-1} u^v = \Gamma(v+1)u, $$

$$ {}_0D_u^{v-2} u^v = \frac{\Gamma(v+1)}{\Gamma(3)} u^2, \dots, {}_0D_u^{v-n} u^v = \frac{\Gamma(v+1)}{\Gamma(n+1)} u^n. $$

The binomial coefficients for $n=0, 1$ and 2 are 1, v and $v(v-1)$. The first several terms of (15) where $f(u)$ is $(1-u)^v$ and $g(u)$ is u^v are

$$ \begin{aligned} {}_0D_u^v(1-u)^v u^v &= \Gamma(v+1)(1-u)^v - v^2\Gamma(v+1)u(1-u)^{v-1} + \\ &\quad + [v(v-1)]^2\Gamma(v+1)u^2(1-u)^{v-2}/(2!)^2 - \dots . \end{aligned} \tag{16} $$

A compact formulation of the above is

$$ {}_0D_u^v u^v(1-u)^v = [\Gamma(v+1)]^3 \sum_{k=0}^{\infty} \frac{(-1)^k u^k (1-u)^{v-k}}{[\Gamma(1+v-k)]^2 (k!)^2}. $$

After replacing u with $(1-z)/2$ in the above, (14) becomes

$$P_\nu(z)=\frac{1}{2^\nu}[\Gamma(\nu+1)]^2\sum_{k=0}^{\infty}\frac{(z-1)^k(z+1)^{\nu-k}}{[\Gamma(1+\nu-k)]^2(k!)^2}. \tag{17}$$

It is easy to verify that the above agrees with the Legendre polynomial $P_n(z)$ when $\nu=n$.

This last result is (9) but was obtained by the means of the fractional calculus. This example again shows the power of fractional calculus and we may expect its influence to be felt in the literature and in the teaching of the mathematical sciences [3].

References

[1] E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, *Cambridge*, 4th edition, 1963, pp. 307, 311.

[2] Lecture Notes in Mathematics, #457, 1974, Fractional Calculus and Its Applications, ed. by Bertram Ross, pp. 23—25.

[3] M. Mikolás, On the Recent Trends in the Development, Theory and Applications of Fractional Calculus, *Ibid*, p. 371.

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