The prime ideals of the Boolean ring of intervals

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In this paper we determine all the prime ideals of the Boolean ring of half-closed intervals of real numbers. Also, as a consequence, we determine all the prime ideals of the Boolean ring of half-closed intervals of rational numbers.

In what follows, B denotes the Boolean ring generated by the set of all left-closed and right-open real intervals. As usual, the set-theoretic symmetric difference and intersection are taken respectively as addition and multiplication in B. Thus, every element of B is a union of finitely many left-closed and right-open real intervals.

For every real number a, we let:

(1) $P(a)=$ *the subring of* B *generated by all the elements of* B *such that* $a \notin [x, y)$

and

(2) $E(a)=$ *the subring of* B *generated by all the elements* $[x, y)$ *of* B *such that* a *is neither a right-end point nor an interior point of* $[x, y)$.

It can readily be verified that $P(a)$ as well as $E(a)$ is a prime ideal of B. Also, it can easily be verified that for every real number a the prime ideals $P(a)$ and $E(a)$ of B are distinct.

We show below that every prime ideal of B is either of type (1) or of type (2).

In what follows (as indicated above) by a prime ideal we mean a proper prime ideal. Also, we recall that in a Boolean ring an ideal is prime if and only if it is maximal. Moreover, if P is an ideal of B then for every element C of B we have:

$$A \in P \quad \text{and} \quad C \subseteq A \quad \text{imply} \quad C \in P. \tag{3}$$

Furthermore, for every element A and H of P we have:

$$(A \cup H) \in P \tag{4}$$

Also, we recall that since $B-P(a)$ is an ultrafilter of B, every element of $B-P(a)$ is an interval $[p, q)$ for some real numbers p and q such that

$$p \leqq a < q. \tag{5}$$

Similarly, every element of $B-E(a)$ is an interval $[t, u)$ for some real numbers t and u such that

$$t < a \leqq u \tag{6}$$

Based on the above notions, we prove the following theorem.

Theorem. *Let P be a proper prime ideal of B. Then one and only one of the following two cases must occur:*

$$P = P(a) \quad or \quad P = E(a)$$

for some real number a.

PROOF. As mentioned above, both cases cannot occur simultaneously. We show below that at least one of the cases must occur.

Since P is a proper (prime) ideal of B, there exists an element $[c, b)$ of B such that

$$[c, b) \notin P. \tag{7}$$

Since $[x, b)$ with $x=b$ is an element (the empty set) of P and since P is an ideal of B, in view of (7) and (3), we see that $\{x|[x, b)\in P\}$ *is a nonempty set of real numbers which is bounded below by c. Let*

$$a = \inf \{x|[x, b) \in P\} \tag{8}$$

for some real number a.

We complete the proof by showing that

$$P = P(a) \quad or \quad P = E(a). \tag{9}$$

Assume the contrary and let

$$P \neq P(a) \quad and \quad P \neq E(a). \tag{10}$$

But then since P, $P(a)$, $E(a)$ are maximal ideals of B, in view of (10), (5), (6), we have:

$$[p, q) \in P \quad with \quad p \leqq a < q \tag{11}$$

and

$$[t, u) \in P \quad with \quad t < a \leqq u \tag{12}$$

for some elements $[p, q)$ and $[t, u)$ of B.

On the other hand, from (8) and (11) it follows that there exists an element $[v, b)$ of B such that

$$[v, b) \in P \quad with \quad a \leqq v < q. \tag{13}$$

But then since P is an ideal of B, in view of (12), (11), (13), (4), we have

$$\big([t, u) \cup [p, q) \cup [v, b)\big) \in P$$

which, in view of (3), implies

$$[t, b) \in P. \tag{14}$$

However, by (12) we have $t<a$ and therefore (14) contradicts (8). Thus, (9) is established and the Theorem is proved.

Remark. We observe that the proof of the Theorem and what precedes it are given in such a language that the statement of the Theorem and its proof remain valid if in them B is replaced by either of B^*, B_0, B_0^*, where:

B^* = the Boolean ring generated by the set of all leftclosed and right-open rational intervals with real end points.

B_0 = the Boolean ring generated by the set of all left-closed and right-open real intervals with rational end points.

B_0^* = the Boolean ring generated by the set of all leftclosed and right-open rational intervals with rational end points.

It is easily seen that each of the Boolean rings B, B^*, B_0, B_0^* is *without unit*, atomless and each has continuumly many prime ideals. Moreover, B as well as B^* is of the power of the contimuum, whereas B_0 as well as B_0^* is denumerable. Clearly, B is isomorphic to B^* and B_0 is isomorphic to B_0^*.

Note. The proof of the Theorem shows that the statement of the Theorem remains valid if it is formulated with respect to any conditionally complete ordered set (instead of the set of all real numbers). However, there is no loss of generality in the present setup of the Theorem.

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